Between Hill evenness and Hill diversity: how fixed-range evenness indices disagree

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Abstract

Ecologists use evenness to describe the distribution of species abundances and compare communities across space and time. Yet the indices in use are not interchangeable. Chao and Ricotta (2019) identified five fixed-range evenness classes built on Hill numbers. At each Hill order α they are values of one normalized Sharma–Mittal family indexed by a scale parameter β. I show that this family is bounded by two rankings: the ranking by Hill evenness D/S ($\beta \to -\infty$) and the ranking by Hill diversity D ($\beta \to +\infty$). For two assemblages at a common order, the indices change their verdict at most once, at a unique β^*, and they do so if and only if D and D/S disagree. The linear and reciprocal classes ($\beta = 0, 2$) therefore decide whether any of the five classes can disagree. The same parameter governs changes of scale. Replicating an assemblage K times drives indices with $\beta < 1$ towards a power of D/S, and indices with $\beta \geq 1$, including Pielou's J, towards one. Along a richness gradient $D \sim cS^{\gamma}$ the limits are 0, γ and 1, so the sign of an evenness–richness trend depends on the index. Published Alpine-succession and Miocene–Pliocene ostracod data show verdicts that reverse with β. The results concern population-level quantities; sampling error is not treated.

Keywords: evenness; Hill numbers; Sharma–Mittal entropy; rank reversal; replication principle; species richness.

1 Introduction

Ecological assemblages differ both in how many species they contain and in how individuals are distributed among those species. Two communities may have the same richness but very different abundance structures: in one, species have broadly comparable relative abundances; in the other, a few species account for most individuals. Evenness describes this aspect of community structure and is used alongside richness and abundance-based diversity to compare habitats, successional stages and assemblages separated in space or time (Hill, 1973; Jost, 2010; Chao and Ricotta, 2019). A claim that one community is more even than another can therefore shape the biological interpretation of dominance patterns, even when the underlying species counts are known.

Measuring this property becomes less straightforward when communities differ in richness. Hill numbers, D_α, express diversity as an effective number of species and use the order α to control the relative contribution of rare and common species (Hill, 1973; Jost, 2010). Evenness indices transform this abundance-based diversity in relation to the number of species, S. There is more than one way to perform that transformation. Two indices may both return a value between zero and one, with one indicating equal abundances, yet assign opposite rankings to communities with different richness. Such a reversal leaves the biological observations unchanged:

it results from the choice of measurement scale. Consequently, an observed difference in evenness across communities, or a trend in evenness along a richness gradient, needs to be distinguished from a conclusion that is robust to the choice of index.

This distinction arises in concrete ecological comparisons. In the Alpine vegetation succession examined by Chao and Ricotta (2019), the abundance structures of communities differ across early, intermediate and late stages of colonization. In coverage-standardized fossil ostracod assemblages compared by Chao et al. (2020), the Miocene and Pliocene differ substantially in both estimated species richness and effective diversity. These examples represent different biological settings, but pose the same interpretive question: when one assemblage is reported to be more even, does that ordering follow from the abundances at the specified Hill order, or does it also depend on the normalization used to express evenness?

The interpretation of evenness is also a subject of current debate. Chisholm (2026) argues that evenness is useful when an index satisfies Lorenz-consistency axioms, and recommends Hill evenness and mean pairwise similarity. Alroy (2025) questions whether evenness should be treated as an ecological property in its own right, notes that Pielou's J does not fully escape dependence on richness and that Hill numbers obeying the replication principle do not measure evenness alone, and proposes fitting abundance-distribution models instead. Independently of which position one adopts, comparisons based on existing indices require clarity about what their normalization does.

Chao and Ricotta (2019) catalogued five classes of evenness measures built on Hill numbers that satisfy a common set of requirements, including a range that does not depend on richness. Pielou's J (Pielou, 1966) is one member. At fixed richness and Hill order these classes are strictly increasing functions of one another, so they rank assemblages identically and differ only in numerical scale; Pierson (2021) studied their nesting and sensitivity at orders 1 and 2. Across different richness values, however, the choice of class can change the verdict. For the ostracod estimates reported by Chao et al. (2020), the linear class and Pielou's J favor the Miocene (0.34 against 0.13, and 0.70 against 0.62), whereas the reciprocal class favors the Pliocene (0.940 against 0.966). It is therefore useful to determine precisely when such an inversion is possible, whether a pair of communities can change rank more than once across normalizations, and what should be reported when indices disagree.

This paper answers these questions exactly. The five classes are values, at each Hill order α, of one function $\beta \mapsto \mathcal{E}_\beta$ of a scale parameter β (Section 2). The main results are the following.

(1) *Two poles.* For two assemblages, the indices of very negative β rank them by Hill evenness D_α/S and those of very positive β rank them by Hill diversity D_α (Proposition 2).

(2) *One crossing.* At a common order, the verdict changes at most once as β varies, at a unique β^*. A reversal exists if and only if D_α and D_α/S rank the assemblages in opposite ways (Theorem 1). The linear and reciprocal classes then give an exact test that covers all five classes of Chao and Ricotta (Corollary 2).

(3) *Geometry.* Along a curve of constant evenness $\mathcal{E}_\beta$, the elasticity of effective diversity with respect to richness equals $\mathcal{E}_{2-\beta}$ (Proposition 3). Replication moves an assemblage parallel to the line of constant D/S, which explains why no finite β is replication-invariant.

(4) *Changes of scale.* Under replication, indices with $\beta < 1$ converge to a power of D/S, whereas those with $\beta \geq 1$, Pielou's J included, converge to one (Theorem 2). Along a richness gradient $D_\alpha \sim cS^\gamma$, the limits are 0, γ and 1 for $\beta < 1$, $\beta = 1$ and $\beta > 1$ (Theorem 3). The sign of an evenness–richness trend is therefore partly a property of the index.

The analysis concerns the interpretation of existing Hill-based indices in ecological comparisons across unequal richness; no new index is proposed. It states when a ranking depends on

the normalization and describes how the indices behave under replication and richness growth. All results concern population-level quantities. Sampling error and the estimation of richness, which Alroy's critique also raises, are treated separately (Delvecchio, 2026) and are not analyzed here.

2 Setting: five classes, one family

2.1 Hill numbers and normalized evenness

Let $p = (p_1, \dots, p_S)$ be a vector of relative abundances with $p_i > 0$ and $\sum_i p_i = 1$, so that $S \geq 2$ is the realized richness. For $\alpha > 0$, $\alpha \neq 1$, the Hill number is $D_\alpha(p) = \left(\sum_i p_i^\alpha\right)^{1/(1-\alpha)}$, and $D_1 = \exp(-\sum_i p_i \ln p_i)$ (Hill, 1973). It satisfies $1 \leq D_\alpha \leq S$, with $D_\alpha = S$ if and only if p is uniform, and it is the number of equally abundant species that would give the same diversity at order α (Jost, 2010). The ratio D_α/S is Hill evenness (Jost, 2010; Tuomisto, 2012). If a fixed universe retains zero-abundance categories, S may be read as capacity instead of realized richness; nothing below changes.

Definition 1 (Normalized Sharma–Mittal evenness)**.** For $S > 1$, $1 \leq D \leq S$ and $\beta \in \mathbb{R}$, put $t = 1 - \beta$ and

$$\mathcal{E}_\beta(D, S) = \frac{D^t - 1}{S^t - 1} \quad (\beta \neq 1), \qquad \mathcal{E}_1(D, S) = \frac{\ln D}{\ln S}. \tag{1}$$

For an assemblage p the evenness of order α and scale β is $\mathcal{E}_{\alpha,\beta}(p) = \mathcal{E}_\beta(D_\alpha(p), S)$.

This is the Sharma–Mittal entropy $\left(D_\alpha^{1-\beta} - 1\right)/(1-\beta)$ (Sharma and Mittal, 1975) divided by its maximum at uniformity. The Hill order α sets how abundances are weighted; β sets the scale on which D_α is compared with S. Equivalently $\mathcal{E}_\beta = \ln_\beta D / \ln_\beta S$ with $\ln_\beta x = (x^{1-\beta} - 1)/(1-\beta)$. The function is continuous in β, takes values in $[0, 1]$, increases strictly in D, decreases strictly in S, and equals 1 only at $D = S$.

2.2 The five classes

Table 1: The five Hill-based evenness classes of Chao and Ricotta (2019) as values of $\mathcal{E}_\beta$ at Hill order α. For $0 < \alpha \leq 2$ all five scale values lie in $[0, 2]$.

Class	Definition	β	Special cases
E_1	$(D_\alpha^{1-\alpha} - 1)/(S^{1-\alpha} - 1)$	α	Pielou ($\alpha = 1$); equals E_4 at $\alpha = 2$
E_2	$(D_\alpha^{\alpha-1} - 1)/(S^{\alpha-1} - 1)$	$2 - \alpha$	Pielou ($\alpha = 1$); equals E_3 at $\alpha = 2$
E_3	$(D_\alpha - 1)/(S - 1)$	0	Heip, 1974 ($\alpha = 1$); Kvålseth, 1991 ($\alpha = 2$)
E_4	$(1 - D_\alpha^{-1})/(1 - S^{-1})$	2	reciprocal class
E_5	$\ln D_\alpha / \ln S$	1	Pielou ($\alpha = 1$)

Table 1 gives the classes (Chao and Ricotta, 2019; the attributions of special cases follow that paper; the path E_1 is attributed there to Mendes et al., 2008). All of them, and in fact every finite β, satisfy the requirements of Chao and Ricotta (Appendix C). The key observation for what follows is simple. Classes E_1 and E_2 change β with α when one moves along an order profile, but a comparison of two assemblages is always made at one order. At a given $\alpha \in (0, 2]$ the five classes are therefore the values of the single function $\beta \mapsto \mathcal{E}_{\alpha,\beta}$ at $\beta \in \{0,\ 1,\ 2,\ \alpha,\ 2-\alpha\}$, and all five lie in the interval $[0, 2]$ spanned by E_3 and E_4. Figure 1 places the five classes in the (β, α) plane. A horizontal slice fixes the Hill order and varies only the normalization; this is the comparison considered in Section 3. The two horizontal slices shown correspond to the Miocene–Pliocene ($\alpha = 1$) and Alpine ($\alpha = 2$) examples analyzed in Section 3.2.

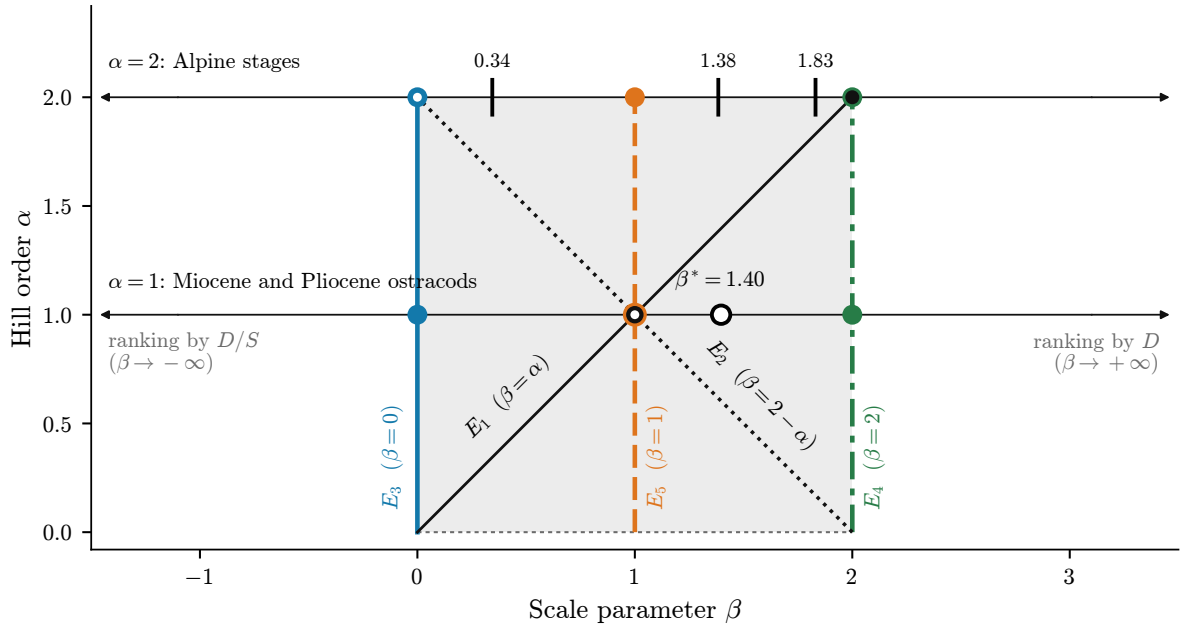


Figure 1: The five Hill-based classes in the (β, α) plane. The fixed-β classes E_3, E_5 and E_4 are vertical lines at $\beta = 0, 1, 2$; the coupled paths E_1 and E_2 follow $\beta = \alpha$ and $\beta = 2 - \alpha$. The canonical range is $0 \leq \beta \leq 2$, $0 < \alpha \leq 2$. The horizontal slices at $\alpha = 1$ (Miocene–Pliocene) and $\alpha = 2$ (Alpine succession) hold Hill order constant. Concentric markers denote coincident classes: $E_1 = E_2 = E_5$ at $\alpha = 1$, and $E_2 = E_3$, $E_1 = E_4$ at $\alpha = 2$. The circled intersection on the $\alpha = 1$ line and the three ticks on the $\alpha = 2$ line locate the crossings β^* computed from the published estimates. The horizontal scale extends beyond the canonical square towards the ranking poles D/S and D, identified as $\beta \to -\infty$ and $\beta \to +\infty$.

2.3 Same richness: one ranking, several scales

Lemma 1 (Fixed richness)**.** (i) Fix S and α. For any two assemblages on S species, every $\mathcal{E}_{\alpha,\beta}$ ranks them identically, since each is a strictly increasing function of D_α. (ii) For $1 < D < S$ the map $\beta \mapsto \mathcal{E}_\beta(D, S)$ is strictly increasing; in particular $E_3 < E_5 < E_4$.

Part (i) is the reason why, at fixed richness, the choice among the classes is a choice of numerical scale. Part (ii) extends the nesting that Pierson (2021) established at orders 1 and 2 to every order and to all real β; the proof is in Appendix B. The scale is not innocuous, as the next statement shows.

Proposition 1 (What a value means)**.** (i) At fixed S, a given value E corresponds to $D = [1 + E(S^t - 1)]^{1/t}$, and to $D = S^E$ at $\beta = 1$. For $S = 100$ and $E = 0.5$ one finds $D = 50.5$, 10 and 1.98 at $\beta = 0, 1, 2$. (ii) For every $S > 1$,

$$\sup_{D} \sup_{\beta_1, \beta_2 \in [0,2]} \left| \mathcal{E}_{\beta_2}(D, S) - \mathcal{E}_{\beta_1}(D, S) \right| = \frac{\sqrt{S} - 1}{\sqrt{S} + 1}, \tag{2}$$

attained by $E_4 - E_3$ at $D = \sqrt{S}$, where $E_5 = 1/2$. (iii) The ratio of the slope of $\mathcal{E}_\beta$ in D at $D = 1$ to its slope at $D = S$ equals S^β.

The same assemblage can thus receive values that differ by $1/3$ at $S = 4$ and by 0.71 at $S = 35$, depending only on which class is used. The local resolution also differs: by (iii), a difference in D near maximal concentration weighs 1, S and S^2 times as much as the same difference near uniformity for E_3, E_5 and E_4. Absolute values, and thresholds for calling an assemblage "even", cannot be transferred from one class to another.

3 Two poles and one crossing

When richness differs, the denominator of $\mathcal{E}_\beta$ changes with the assemblage and Lemma 1(i) no longer applies. The next statement identifies the two orderings between which every finite β lies.

Proposition 2 (The two poles)**.** Fix $\alpha > 0$ and an assemblage with $1 < D_\alpha < S$. Then

$$\mathcal{E}_\beta = \Big(\frac{D_\alpha}{S}\Big)^{1-\beta}[1+o(1)] \quad (\beta \to -\infty), \qquad 1 - \mathcal{E}_\beta = D_\alpha^{1-\beta}[1+o(1)] \quad (\beta \to +\infty). \tag{3}$$

Hence, for two assemblages whose D_α/S differ, sufficiently negative β ranks them by D_α/S, and for two assemblages whose D_α differ, sufficiently positive β ranks them by D_α.

Neither pole is an evenness index in the sense of Chao and Ricotta: D/S has a minimum that depends on S, and D is a diversity. They are the limits of ordering between which the fixed-range indices lie. The first is Hill evenness, a replication-invariant ratio. The second is the richness-carrying quantity that doubles when an assemblage is replicated twice.

Theorem 1 (Single crossing)**.** Fix a common order $\alpha > 0$ and two assemblages A, B with $1 < D_A < S_A$ and $1 < D_B < S_B$, where $D = D_\alpha$. Let $\Delta(\beta) = \mathcal{E}_\beta(A) - \mathcal{E}_\beta(B)$. Then $\Delta \equiv 0$ if and only if $(D_A, S_A) = (D_B, S_B)$. Otherwise Δ has at most one zero on $\mathbb{R}$, and it has exactly one if and only if

$$(D_A - D_B)\Big(\frac{D_A}{S_A} - \frac{D_B}{S_B}\Big) < 0. \tag{4}$$

At that zero, β^*, the difference changes sign (the crossing is transversal).

In words: if the assemblage with the higher effective diversity also has the higher Hill evenness, no change of scale can reverse their ranking; if the two quantities disagree, there is exactly one scale at which the ranking turns. The proof (Appendix A) writes $\ln \mathcal{E}_\beta$ as an integral of a kernel and shows that the ratio of two such integrals is strictly monotone in β, with limits fixed by the two poles. The theorem and Proposition 2 give the direction of the reversal.

Corollary 1 (Unanimity and the two sides of β^*)**.** Let $(D_A, S_A) \neq (D_B, S_B)$ as in Theorem 1. (i) If $D_A \geq D_B$ and $D_A/S_A \geq D_B/S_B$, then $\mathcal{E}_\beta(A) > \mathcal{E}_\beta(B)$ for every β. (ii) If (4) holds, then $\Delta(\beta)$ has the sign of $D_A/S_A - D_B/S_B$ for $\beta < \beta^*$ and the sign of $D_A - D_B$ for $\beta > \beta^*$.

Concordant pole rankings guarantee a common strict ranking for every real β. If the poles rank the assemblages oppositely, the curves cross once on $\mathbb{R}$, but the five classes can still agree when β^* lies outside $[0, 2]$. The next corollary gives the exact test for the canonical interval.

Corollary 2 (Exact test for the five classes)**.** Fix $0 < \alpha \leq 2$ and write $\Delta_3 = \mathcal{E}_0(A) - \mathcal{E}_0(B)$, $\Delta_4 = \mathcal{E}_2(A) - \mathcal{E}_2(B)$.

(i) If $\Delta_3\Delta_4 > 0$, the five classes $E_1, \ldots, E_5$ give the same verdict at order α, namely the common sign (and so does every β in $[0, 2]$).

(ii) If $\Delta_3\Delta_4 < 0$, there is exactly one $\beta^* \in (0, 2)$; E_3 follows the ranking by D/S and E_4 the ranking by D. Each of E_1, E_2 and E_5 follows E_3 if its scale (α, $2 - \alpha$ and 1, respectively) is below β^*, and E_4 if it is above.

(iii) If exactly one of Δ_3, Δ_4 vanishes, β^* is at that endpoint.

More generally, for any $\beta_1 < \beta_2$ the curves cross in $[\beta_1, \beta_2]$ if and only if $\Delta(\beta_1)\Delta(\beta_2) \leq 0$. Moreover $\beta^* = 1$ if and only if $\ln D_A / \ln S_A = \ln D_B / \ln S_B$.

Proof. By Theorem 1 the sign of Δ changes at most once, and only transversally; hence $\Delta(\beta_1)\Delta(\beta_2) > 0$ means that no zero lies in $[\beta_1, \beta_2]$, and < 0 that exactly one does. For $0 < \alpha \leq 2$ the scales α, $2 - \alpha$, 1 lie in $[0, 2]$. Case (ii) uses Corollary 1. Both endpoint values cannot vanish for nonidentical curves. At $\beta = 1$, $\mathcal{E}_1(A) = \mathcal{E}_1(B)$ is the stated equality. □

The test requires nothing beyond D_α and S of each assemblage. Figure 1 shows why the endpoint test covers all five classes: at a fixed $0 < \alpha \leq 2$, their scale values lie between $\beta = 0$ and $\beta = 2$. When $0 < \beta^* < 2$, classes on opposite sides of the crossing give opposite rankings; when $\beta^* \notin [0, 2]$, all five classes agree. Endpoint crossings are covered by Corollary 2(iii). The script supplied with this paper computes β^* and the verdicts of E_3, E_5 and E_4 from these four numbers.

3.1 A geometric reading

Let $x = \ln S$ and $y = \ln D$. Since replication multiplies both D and S by K, it shifts a point along the diagonal $y = x + \text{const}$, on which D/S is constant. The following identity describes the curves of constant evenness in this plane.

Proposition 3 (Curves of constant evenness)**.** For $1 < D < S$ and every real β, along a curve $\mathcal{E}_\beta(D, S) = \text{const}$,

$$\left.\frac{\partial \ln D}{\partial \ln S}\right|_{\mathcal{E}_\beta} = -\frac{S\,\partial_S \mathcal{E}_\beta}{D\,\partial_D \mathcal{E}_\beta} = \mathcal{E}_{2-\beta}(D, S). \tag{5}$$

Consequently: (a) the slope of the curve lies in $(0, 1)$ and decreases strictly in β, from 1 as $\beta \to -\infty$ to 0 as $\beta \to +\infty$, and equals $\ln D / \ln S$ at $\beta = 1$; (b) under K-fold replication

$$\frac{\mathrm{d}\mathcal{E}_\beta}{\mathrm{d} \ln K} = D\,\partial_D \mathcal{E}_\beta \left(1 - \mathcal{E}_{2-\beta}\right) > 0; \tag{6}$$

(c) for a fixed value $e \in (0, 1)$ and $S \to \infty$, the curve $\mathcal{E}_\beta = e$ satisfies $D \sim e^{1/(1-\beta)} S$ if $\beta < 1$, $D = S^e$ if $\beta = 1$, and $D \to (1 - e)^{-1/(\beta-1)}$ if $\beta > 1$.

Identity (5) says that the local trade-off between richness and effective diversity along a curve of constant $\mathcal{E}_\beta$ is the value of the reflected index $\mathcal{E}_{2-\beta}$; the point $\beta = 1$ is self-dual, and E_3 and E_4 are each other's reflection. Part (a) is the mechanism behind Theorem 1 (Figure 2a): as β grows, the curves through a point rotate from the direction of constant D/S to the direction of constant D, and the assemblages that can be reached by such a rotation are those in the two wedges between these directions, which is condition (4). Part (b) shows that no member with finite β is invariant to replication: invariance would require curves of slope one, and $\mathcal{E}_{2-\beta} < 1$ for every nonuniform assemblage. Part (c) gives the standard of constant evenness imposed by each scale; it is the counterpart of Theorem 3 below.

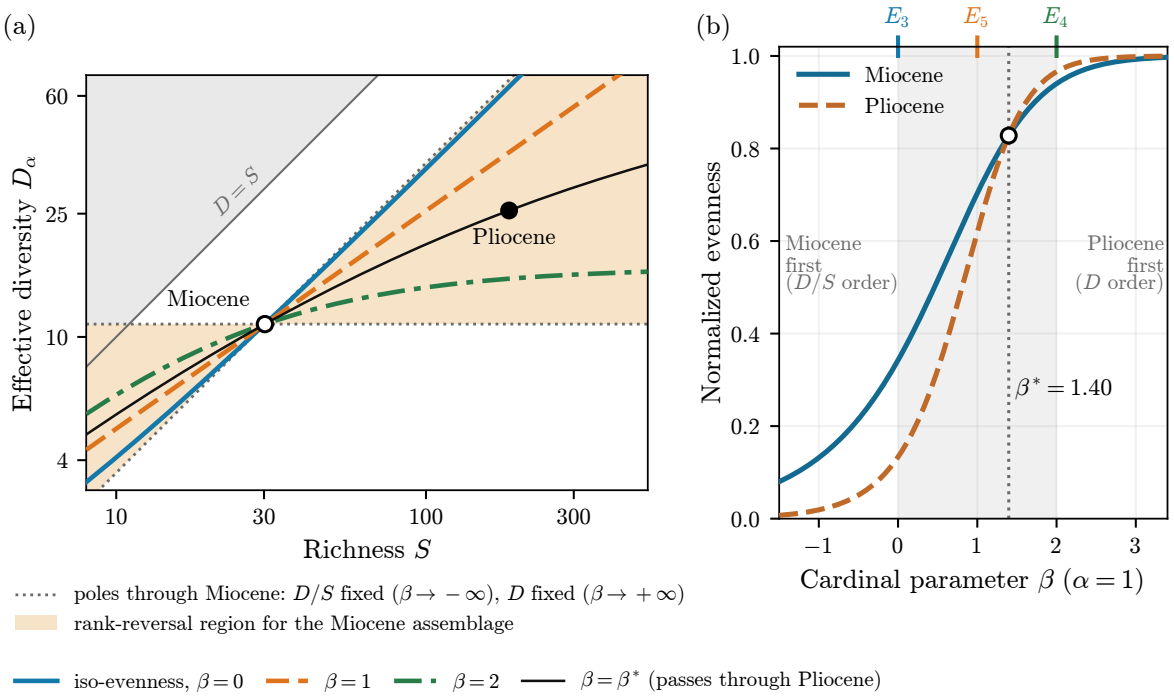


Figure 2: Single crossing for the Miocene and Pliocene ostracod assemblages (Chao et al., 2020; $\alpha = 1$; Miocene $(S, D_1) = (30.2, 11.0)$, Pliocene $(185.5, 25.6)$). (a) The plane of richness and effective diversity, both on logarithmic axes. Curves through the Miocene point show where a given $\mathcal{E}_\beta$ stays constant; their slope falls from that of the diagonal (D/S fixed) to zero (D fixed) as β increases (Proposition 3). An assemblage in the shaded wedges is ranked differently by the two poles; the Pliocene point lies on the curve with $\beta = \beta^*$. The gray region is $D > S$. (b) The corresponding evenness curves cross once. Lines are functions of the published point estimates, not confidence bands.

3.2 Two published data sets

Miocene and Pliocene ostracods. Chao et al. (2020) compare Miocene and Pliocene assemblages at a common coverage of 99.3%. Their standardized richness and Shannon diversity are $(S, D_1) = (30.2, 11.0)$ for the Miocene and $(185.5, 25.6)$ for the Pliocene. Since standardized richness is an estimate and not an integer, $\mathcal{E}_\beta$ is evaluated as a function of (D, S) and not as an index of a probability vector. Table 2 gives the three sections. The Pliocene has the higher diversity, $D_1 = 25.6 > 11.0$, but the lower Hill evenness, $D_1/S = 0.138 < 0.364$. The two poles disagree, so condition (4) holds and there is exactly one reversal, at $\beta^* = 1.3966$. At $\alpha = 1$, the crossing lies between the E_5 and E_4 paths in Figure 1: E_3 and E_5, with scales below β^*, favor the Miocene, whereas E_4 favors the Pliocene. Figure 2b displays the corresponding evenness curves. The reversal at E_4 is small, $\mathcal{E}_2(\text{Miocene}) - \mathcal{E}_2(\text{Pliocene}) = -0.026$, and so is its sensitivity: letting each published value vary by ± 0.05 moves β^* within $[1.3865, 1.4067]$ and leaves this difference between -0.0265 and -0.0254. These are arithmetic checks, not confidence bounds. The conclusion that the Miocene assemblage is the more even depends on a scale below 1.40. The column labelled $q = 2$ in the same table of Chao et al. is a different quantity, $\mathcal{E}_{2,0}$: it changes the Hill order and keeps $\beta = 0$, whereas the comparison here fixes $\alpha = 1$ and changes β.

Alpine succession. For the public succession data analyzed by Chao and Ricotta (2019) I take $\alpha = 2$; at this order $E_2 = E_3$ and $E_1 = E_4$, so there are three distinct scales. Table 3 gives the three stages. The ranking by D/S (and by E_3) is late > early > mid, and the ranking by D_2 (and by E_4) is mid > early > late. The three stages therefore pass from one order to

Table 2: Miocene and Pliocene assemblages at maximal common coverage (published estimates in Chao et al., 2020, Table 2, rounded to one decimal) and three evenness classes at $\alpha = 1$.

Assemblage	S	D_1	D_1/S	E_3	E_5	E_4
Miocene	30.2	11.0	0.364	0.3425	0.7036	0.9402
Pliocene	185.5	25.6	0.138	0.1333	0.6208	0.9661

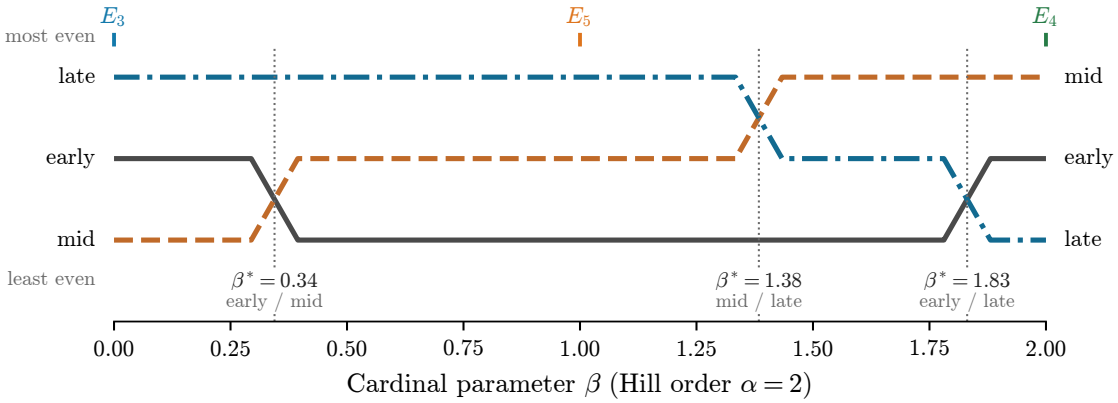


Figure 3: Order of the three Alpine succession stages across the scale parameter at $\alpha = 2$. Each pair of stages exchanges position exactly once; the three exchanges sit at the values of β^* shown. The order at $\beta = 0$ is that of D_2/S and the order at $\beta = 2$ is that of D_2. Transitions are drawn schematically over a short interval.

the other. By Theorem 1 each pair crosses exactly once, at $\beta^* = 0.344$ (early and mid), 1.384 (mid and late) and 1.831 (early and late). These are the three ticks along the $\alpha = 2$ slice of Figure 1; Figure 3 shows how they change the ranks of the three stages. At each pair, at least one endpoint gap is below 0.005; the reversals are ordinally genuine but numerically small, and their ecological relevance cannot be judged from point estimates. Two different quantities are involved. Within each stage the spread $E_4 - E_3$ is 65.4%, 64.5% and 60.1% of the bound (2) (early, mid, late), and measures the effect of the scale. The gap between stages measures how fragile a ranking is, and it is much smaller.

Table 3: Alpine succession stages at $\alpha = 2$ (richness and D_2 from the data of Chao and Ricotta, 2019).

Stage	S	D_2	D_2/S	E_3	E_5	E_4
Early	35	18.28	0.522	0.5082	0.8173	0.9731
Mid	41	21.23	0.518	0.5058	0.8228	0.9767
Late	27	15.70	0.581	0.5654	0.8355	0.9723

4 When richness changes: replication and gradients

The poles also govern what happens when richness itself changes. I consider two stylized settings: replication, in which an assemblage is copied, and a richness gradient in which effective diversity grows as a power of richness.

4.1 Replication

Take K disjoint replicas of an assemblage, each with its own species and the same relative abundances. Then $D_\alpha \to KD_\alpha$ and $S \to KS$ at every order. Hill evenness D_α/S is unchanged. A Lorenz-based comparison (Gosselin, 2001), which looks at cumulative shares, is also unchanged because the replicated assemblage has the same Lorenz curve. The fixed-range classes are not unchanged.

Theorem 2 (Replication response)**.** Let $1 < D < S$ and denote by $\mathcal{E}_\beta^{(K)} = \mathcal{E}_\beta(KD, KS)$ the evenness of the K-fold replicate. For every finite β it increases strictly in K, and as $K \to \infty$:

(i) if $\beta < 1$, then $\mathcal{E}_\beta^{(K)} \to (D/S)^{1-\beta}$, with an error of order $K^{-(1-\beta)}$;

(ii) if $\beta = 1$, then $1 - \mathcal{E}_1^{(K)} = \ln(S/D)/\ln(KS) \sim \ln(S/D)/\ln K$;

(iii) if $\beta > 1$, then $1 - \mathcal{E}_\beta^{(K)} \sim \big(D^{1-\beta} - S^{1-\beta}\big)K^{-(\beta-1)}$.

Monotonicity follows from (6); the limits and rates follow from the closed form (1) (Appendix B). For $\beta < 1$ the limit is a strictly increasing function of Hill evenness, so replicated assemblages are asymptotically ordered by D/S. At $\beta = 0$ the limit is D/S itself. For $\beta \geq 1$ the index converges to one, whatever the assemblage: the information survives only in the rate, and for $\beta = 1$ the leading term again ranks by D/S. Pielou's J is the borderline case. A community copied a million times has $J = 0.94$ for the Miocene values of Table 2, against $J = 0.70$ at $K = 1$, and a reciprocal-class value of 0.9999999 (Figure 4a). A property that is desirable in a measure of evenness, that copying an assemblage does not change the measure, is therefore lost gradually when $\beta < 1$ and almost completely when $\beta \geq 1$.

4.2 Richness gradients

Consider a sequence of assemblages indexed by richness, with effective diversity growing as

$$D_\alpha(S) \sim c\, S^\gamma, \qquad c > 0,\ 0 \leq \gamma \leq 1, \tag{7}$$

where the constants are compatible with $1 \leq D_\alpha \leq S$. The exponent γ is the growth rate of effective diversity relative to richness. This is a stylized scaling, not a claim about real assemblages; it is used to ask what each index does when S varies.

Theorem 3 (Richness-gradient limits)**.** Under (7), as $S \to \infty$:

(i) if $\beta < 1$ and $0 < \gamma < 1$, then $\mathcal{E}_\beta \sim c^{1-\beta} S^{-(1-\gamma)(1-\beta)} \to 0$; if $\gamma = 1$, then $\mathcal{E}_\beta \to c^{1-\beta}$; if $\gamma = 0$ and $D_\alpha \to c < \infty$, then $\mathcal{E}_\beta \to 0$;

(ii) if $\beta = 1$, then $\mathcal{E}_1 \to \gamma$;

(iii) if $\beta > 1$ and $\gamma > 0$, then $\mathcal{E}_\beta \to 1$; if $\gamma = 0$ and $D_\alpha \to c < \infty$, then $\mathcal{E}_\beta \to 1 - c^{1-\beta}$.

The three regimes impose three different standards of evenness along a gradient; Proposition 3(c) restates them for a constant value. For $\beta < 1$, positive asymptotic evenness requires effective diversity to grow in proportion to richness. At $\beta = 1$ the index records the exponent γ itself. For $\beta > 1$, any unbounded growth of effective diversity eventually looks perfectly even. Table 4 collects these limits together with Theorem 2.

The practical consequence is shown in Figure 4b. Over $10 \leq S \leq 1000$ with $D_\alpha = 0.7\, S^{0.8}$, the same sequence of assemblages gives E_3 falling from 0.38 to 0.18, E_5 rising from 0.65 to 0.75 and E_4 rising from 0.86 to 0.995. Three indices of the same fixed-range family yield a negative, a positive and a saturating relationship between evenness and richness for the same communities. No sampling is involved: the dependence on richness is a property of the normalization.

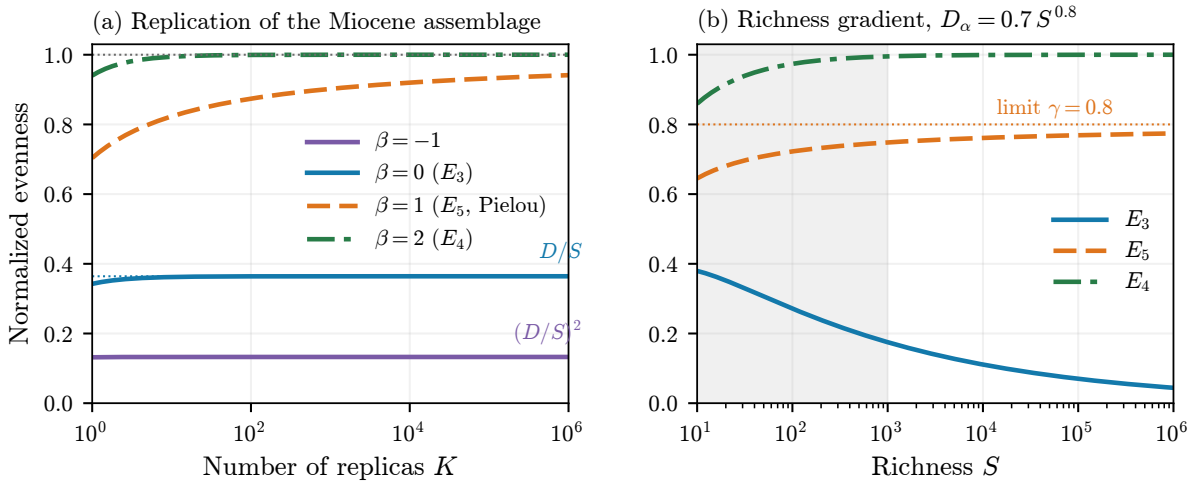


Figure 4: Behavior of three classes at order α under changes of scale. (a) Replication of the Miocene assemblage ($S = 30.2$, $D_1 = 11.0$): E_3 converges to $D/S = 0.364$, E_5 (Pielou's J) and E_4 converge to one. The purple curve ($\beta = -1$, outside the five classes) converges to $(D/S)^2$. (b) A richness gradient $D_\alpha = 0.7\,S^{0.8}$: E_3 falls, E_5 tends to $\gamma = 0.8$ and E_4 rises towards one. The shaded band marks $10 \le S \le 1000$.

Table 4: Behavior of $\mathcal{E}_\beta$ when richness changes. K: number of replicas; $S \to \infty$ along $D_\alpha \sim cS^\gamma$.

Scale	Replication, $K \to \infty$	Gradient, $S \to \infty$	Constant $\mathcal{E}_\beta = e$ requires
$\beta < 1$	$(D/S)^{1-\beta}$; error $O(K^{-(1-\beta)})$	0 if $\gamma < 1$; $c^{1-\beta}$ if $\gamma = 1$	$D \sim e^{1/(1-\beta)}S$
$\beta = 1$	$1 - \mathcal{E} \sim \ln(S/D)/\ln K$	γ	$D = S^e$
$\beta > 1$	$1 - \mathcal{E} \sim (D^{1-\beta} - S^{1-\beta})K^{1-\beta}$	1 if $\gamma > 0$	D bounded

5 Discussion

5.1 Ecological interpretation of index-dependent rankings

An index-dependent ranking must be interpreted differently from a change in the observed composition of an assemblage. In the ostracod comparison, the coverage-standardized Pliocene estimate has greater Shannon effective diversity, D_1, but a smaller D_1/S than the Miocene estimate. The linear and logarithmic classes consequently rank the Miocene higher, while the reciprocal class ranks the Pliocene higher. These are alternative numerical assessments of the same published diversity and richness estimates. The reversal does not, by itself, establish a different biological history for either period, nor does it identify a biological mechanism responsible for the difference in species abundances.

The Alpine succession example raises the same issue within a temporal ecological sequence. The three stages can be ordered differently by the canonical evenness classes at the same Hill order, even though the stage abundances used for the calculation do not change. An interpretation of succession as moving towards greater or lesser evenness is therefore conditional on the chosen index unless the rankings agree. Moreover, the small between-stage gaps in this example indicate that sampling uncertainty could matter to any empirical claim of a difference. The crossing theorem concerns the published point estimates; it does not establish statistical significance.

These results provide a way to separate two questions that ecological reports sometimes combine. One is whether the effective number of species is larger; the other is how that

effective number compares with the richness of the assemblage. Neither question is intrinsically a substitute for the other. Agreement between the linear and reciprocal classes supports a ranking that is robust across all five Hill-based classes considered here, at the specified order. Disagreement identifies a dependence on the normalization and calls for reporting D_α, S and the relevant scale parameter. Inferences about colonization, environmental filtering, disturbance or other ecological processes require evidence beyond an evenness ranking.

5.2 What the results say to each position

Chao and Ricotta (2019) and Pierson (2021). The five classes were derived and compared at fixed richness, where they differ only in scale. The results above show where they stop being interchangeable: when richness differs, and a reversal exists exactly when D and D/S disagree. The question "which of the five classes should be used" becomes a question about which pole the comparison is meant to follow, and Corollary 2 tells the user at once whether the question matters for a given pair of assemblages. Reversals induced by the Hill order are classical in diversity profiles (Patil and Taillie, 1982; Tóthmérész, 1995); the present results are the counterpart along the second axis of the Sharma–Mittal plane, where at a fixed order a change of scale can also reverse a ranking, but only once.

Chisholm (2026). One of the two indices he recommends, Hill evenness, is the D/S pole of this family. Lorenz-based comparisons (Gosselin, 2001) are unaffected by replication, because a replicated assemblage has the same Lorenz curve; D/S has this property and no member of the fixed-range family with finite β does (Proposition 3(b)). Theorem 2 grades the failure. Indices with $\beta < 1$ approach a monotone function of D/S and so become asymptotically order-equivalent to Hill evenness; indices with $\beta \geq 1$, including Pielou's J, saturate at one. Within the Hill-based family studied here, the Lorenz-consistent ratio D/S defines the negative-β ranking pole. Other Lorenz-consistent measures, including mean pairwise similarity, need not belong to this family. The fixed-range classes correspond to finite β and retain a dependence on richness quantified by the results above. Whether a fixed range or replication invariance matters more is a modeling decision and not a mathematical one; what the paper adds is that it can be quantified.

Alroy (2025). His observations that Pielou's J does not fully escape richness and that Hill numbers carry richness have exact counterparts for population-level quantities: D is the richness-carrying pole, and the fixed-range classes inherit a dependence on S from the normalization. Theorems 2 and 3 specify that dependence. An empirical correlation between evenness and richness therefore cannot be interpreted without naming the scale: the same communities can give a falling, a rising or a saturating relationship (Figure 4b). This isolates the deterministic part of the entanglement. It does not address the sampling component of the critique, namely that singletons and undersampled richness distort observed indices, nor his recommendation to fit abundance models; those require the estimation theory cited in Section 5.4.

5.3 A reporting rule

The results suggest a rule that costs little and removes the ambiguity.

1. Report α, S and D_α for each assemblage, not only an evenness value.
2. Compute E_3 and E_4. If both rank the assemblages in the same way, the verdict is the same for all five classes at orders $0 < \alpha \leq 2$ (Corollary 2) and can be reported as independent of the scale.
3. If they disagree, state that the comparison is a question about D against D/S, say which of the two the study is about, and report β^*.
4. When evenness is related to richness, or assemblages of different extent are pooled, name

the index; the sign of the relationship can come from β (Theorem 3). If invariance to replication is wanted, use D/S.

5. Do not compare values across classes; Proposition 1 converts a value into an effective number.

5.4 Scope and limits

The results concern population-level D_α and S. In practice both are estimated, richness being the harder one, and a reversal that is real at the population level may not be detectable from samples; the coverage-standardized values used here are point estimates, and the ± 0.05 check above is arithmetic, not inference. Bias and joint error in evenness estimation are analyzed in Delvecchio (2026), and a smooth monotone transformation rescales a local signal and its standard error by the same factor, so a larger derivative does not mean greater inferential precision.

The theorem is stated at a common α. Comparisons of order profiles for E_1 and E_2, in which α and β vary together, combine the two axes and are not covered; at each single order, however, their verdicts are determined as in Corollary 2. For $\alpha > 2$ the scales of E_1 and E_2 leave $[0, 2]$, and the interval version of the test applies with the appropriate endpoints.

Two measures lie outside the family and delimit it (Appendix D). The Euclidean Minkowski evenness of Chao and Ricotta is a function of D_2 alone but not a member, because its slope at uniformity diverges. Chisholm's mean pairwise similarity $1 - G$ is not a function of D_2: two four-species assemblages with the same $D_2 = 200/67$ have $1 - G = 0.675$ and 0.725. Theorem 1 applies to neither.

6 Conclusions

At each Hill order, the five fixed-range evenness classes of Chao and Ricotta are points of one scale parameter between two rankings, that of Hill evenness D/S and that of Hill diversity D. If D and D/S rank two assemblages oppositely, their evenness curves cross exactly once as β varies over $\mathbb{R}$; otherwise no crossing occurs. Whether this crossing falls within the range of the five canonical classes is decided by the linear and reciprocal indices at $\beta = 0$ and $\beta = 2$. Along the same scale parameter, replication and richness gradients move evenness towards D/S, one, or an exponent, so that the apparent relationship between evenness and richness depends on the index. Evenness comparisons across unequal richness are then comparisons between D and D/S, and reporting both quantities, with β^*, makes the dependence on the index explicit. In ecological applications, this distinguishes rankings that are stable across the five classes from conclusions about dominance or succession that depend on the chosen normalization.

A Proof of Theorem 1

Write $a = \ln D_A$, $b = \ln S_A$, $c = \ln D_B$, $d = \ln S_B$, so $0 < a < b$ and $0 < c < d$. Put $t = 1 - \beta$ and define the strictly positive smooth kernel

$$\phi_t(x) = \begin{cases} \dfrac{t}{1 - \mathrm{e}^{-tx}}, & t \neq 0, \\ \dfrac{1}{x}, & t = 0, \end{cases} \qquad x > 0.$$

The kernel is smooth across $t = 0$ since $\phi_t(x) = B(tx)/x$ with $B(y) = y/(1 - \mathrm{e}^{-y})$, so differentiation in t under the integral sign over compact subintervals of $(0, \infty)$ is justified. Direct

integration gives

$$\ln \mathcal{E}_{1-t}(D,S) = -\int_{\ln D}^{\ln S} \phi_t(x)\,\mathrm{d}x,$$

including $t = 0$, where both sides equal $\ln(\ln D/\ln S)$. The logarithmic derivative $h_t(x) = \partial_t \ln \phi_t(x)$ is strictly increasing in x: for $t \neq 0$,

$$\partial_x h_t(x) = \frac{1 + (tx-1)\mathrm{e}^{tx}}{(\mathrm{e}^{tx}-1)^2} > 0, \qquad \partial_x h_0(x) = \tfrac{1}{2},$$

because $1 + (y-1)\mathrm{e}^y > 0$ for $y \neq 0$ (it vanishes at zero and has derivative $y\mathrm{e}^y$).

If $(a,b) = (c,d)$ the curves coincide. If only one of D, S differs, the strict monotonicity of $\mathcal{E}_\beta$ in D and in S fixes the order for every β, and the product in (4) is nonnegative. After interchanging labels suppose $a < c$. If $b \geq d$, then $D_A < D_B$ and $S_A \geq S_B$ give $\mathcal{E}_\beta(A) < \mathcal{E}_\beta(B)$ for all β, again with a nonnegative product. The remaining case is $a < c$ and $b < d$. Define

$$I_-(t) = \int_a^c \phi_t\,\mathrm{d}x, \qquad I_+(t) = \int_b^d \phi_t\,\mathrm{d}x, \qquad R(t) = \frac{I_+(t)}{I_-(t)}.$$

Then $\ln \mathcal{E}_{1-t}(A) - \ln \mathcal{E}_{1-t}(B) = I_+(t) - I_-(t) = I_-(t)[R(t)-1]$, and both integrals are strictly positive.

Give each interval the probability density proportional to ϕ_t. The distribution on $[b,d]$ strictly dominates the one on $[a,c]$ in the first-order stochastic sense, since both endpoints lie to the right. If the intervals overlap ($b < c$), for $x \in [b,c]$ write $u = \int_a^b \phi_t > 0$, $v = \int_b^x \phi_t \geq 0$, $w = \int_x^c \phi_t \geq 0$ and $z = \int_c^d \phi_t > 0$; the distribution functions satisfy $F_{[b,d]}(x) = v/(v+w+z) < (u+v)/(u+v+w) = F_{[a,c]}(x)$. Outside the overlap the inequality follows from the order of the endpoints. Integration by parts over $[a,d]$ gives

$$\int h_t\,\mathrm{d}F_{[b,d]} - \int h_t\,\mathrm{d}F_{[a,c]} = \int_a^d (F_{[a,c]} - F_{[b,d]})\,\partial_x h_t\,\mathrm{d}x > 0,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln R(t) = \frac{\int_b^d h_t\phi_t\,\mathrm{d}x}{I_+(t)} - \frac{\int_a^c h_t\phi_t\,\mathrm{d}x}{I_-(t)} > 0.$$

As $t \to -\infty$, $\phi_t(x) \sim |t|\mathrm{e}^{tx}$ uniformly on these intervals, so $R(t) \to 0$ because $b > a$. As $t \to +\infty$, $\phi_t(x) \sim t$ uniformly and $R(t) \to (d-b)/(c-a)$. By strict monotonicity, $R(t) = 1$ has exactly one finite solution if and only if $d - b > c - a$, which is $D_A/S_A > D_B/S_B$. Together with $D_A < D_B$ this is (4); if the limiting ratio is at most one, $R < 1$ for every finite t and there is no root. At a root t^* one has $R(t^*) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\ln \mathcal{E}(A) - \ln \mathcal{E}(B)) = I_-(t^*)R'(t^*) > 0$. Since $t = 1 - \beta$ and the common value at the root is positive, Δ has a nonzero derivative in β there. □

B Proofs of the remaining statements

Lemma 1. Part (i) holds because, for fixed S and β, (1) is a strictly increasing function of D. For (ii), fix $1 < D < S$ and put $r = \ln D/\ln S \in (0,1)$, $u = S^{1-\beta} > 0$. Then $D^{1-\beta} = u^r$ and $\mathcal{E}_\beta = (u^r - 1)/(u-1)$, with value r at $u = 1$. For $0 < r < 1$ the map $u \mapsto u^r$ is strictly concave, so the chord slope from $(1,1)$ to (u,u^r) decreases strictly in u. Since u decreases strictly in β, $\mathcal{E}_\beta$ increases strictly in β.

Proposition 1. (i) is the inversion of (1); for $S = 100$, $E = 0.5$: $1 + 0.5 \cdot 99 = 50.5$; $100^{0.5} = 10$; $(1 + 0.5(100^{-1} - 1))^{-1} = 1/0.505 = 1.98$. (ii) By Lemma 1(ii) the largest difference at fixed D is $\mathcal{E}_2 - \mathcal{E}_0 = (D-1)(S-D)/[D(S-1)] = [S+1-D-S/D]/(S-1)$. Its derivative vanishes only at $D = \sqrt{S}$, where the second derivative is negative, and the value there is $(\sqrt{S}-1)^2/(S-1) = (\sqrt{S}-1)/(\sqrt{S}+1)$. Finally $\ln\sqrt{S}/\ln S = 1/2$. (iii) $\partial_D \mathcal{E}_\beta = (1-\beta)D^{-\beta}/(S^{1-\beta}-1)$ for $\beta \neq 1$ and $1/(D\ln S)$ for $\beta = 1$; in both cases the factor in D is $D^{-\beta}$, which equals 1 at $D = 1$ and $S^{-\beta}$ at $D = S$.

Proposition 2. For $t = 1-\beta \to +\infty$, $\mathcal{E} = (D^t-1)/(S^t-1) = (D/S)^t(1-D^{-t})/(1-S^{-t})$ and the last factor tends to one. For $t \to -\infty$, $1-\mathcal{E} = (D^t - S^t)/(1-S^t) = D^t\big(1-(S/D)^t\big)/(1-S^t)$ and again the last factor tends to one since $1 < D < S$. The ranking statements follow because $(D_A/S_A)^t > (D_B/S_B)^t$ for large t if and only if $D_A/S_A > D_B/S_B$, and $D_A^t < D_B^t$ for large negative t if and only if $D_A > D_B$.

Corollary 1. Part (i): if either inequality is strict, Theorem 1 excludes a crossing (the product in (4) is nonnegative), so Δ keeps one sign for all β; it equals the sign of $D_A/S_A - D_B/S_B$ for $\beta \to -\infty$ and of $D_A - D_B$ for $\beta \to +\infty$ whenever these are nonzero, and at least one of them is. Part (ii): Δ changes sign exactly once, and its signs at the two ends are those of the poles (Proposition 2).

Proposition 3. For $t \neq 0$,

$$\partial_D \mathcal{E}_\beta = \frac{t\,D^{t-1}}{S^t - 1}, \qquad \partial_S \mathcal{E}_\beta = -\frac{t\,S^{t-1}(D^t-1)}{(S^t-1)^2},$$

so $-S\partial_S\mathcal{E}/(D\partial_D\mathcal{E}) = S^t(D^t-1)/\big(D^t(S^t-1)\big) = (1-D^{-t})/(1-S^{-t}) = \mathcal{E}_{2-\beta}$, because $1-(2-\beta) = -t$. For $t = 0$, $\partial_D\mathcal{E}_1 = 1/(D\ln S)$ and $\partial_S\mathcal{E}_1 = -\ln D/(S\ln^2 S)$ give the ratio $\ln D/\ln S$. (a) $\mathcal{E}_{2-\beta} \in (0,1)$ for $1 < D < S$ and is strictly decreasing in β by Lemma 1(ii); its limits are 1 at $\beta \to -\infty$ and 0 at $\beta \to +\infty$ by Proposition 2. (b) With $D \to KD$, $S \to KS$, $\mathrm{d}\mathcal{E}/\mathrm{d}\ln K = D\partial_D\mathcal{E} + S\partial_S\mathcal{E} = D\partial_D\mathcal{E}\,(1-\mathcal{E}_{2-\beta})$, evaluated at the replicated point, which is positive because $\partial_D\mathcal{E} > 0$ and $\mathcal{E}_{2-\beta} < 1$. (c) Solving $\mathcal{E}_\beta = e$ gives $D = [1+e(S^t-1)]^{1/t}$ for $t \neq 0$ and $D = S^e$ for $t = 0$. For $t > 0$ the bracket is $\sim eS^t$, so $D \sim e^{1/t}S$; for $t < 0$, $S^t \to 0$ and $D \to (1-e)^{1/t}$.

Theorem 2. Monotonicity is (6) integrated over K. For $t = 1-\beta > 0$ write $\mathcal{E}^{(K)} = (D^t - K^{-t})/(S^t - K^{-t})$; then $\mathcal{E}^{(K)} - (D/S)^t = -K^{-t}(S^t - D^t)/[S^t(S^t-K^{-t})] = O(K^{-t})$. For $\beta = 1$, $1-\mathcal{E}^{(K)} = \big(\ln(KS) - \ln(KD)\big)/\ln(KS)$. For $\beta > 1$, put $r = \beta - 1$: $1-\mathcal{E}^{(K)} = K^{-r}(D^{-r}-S^{-r})/(1-K^{-r}S^{-r})$.

Theorem 3. Substitute (7) into (1). For $t > 0$ and $\gamma > 0$ both powers diverge and $\mathcal{E} \sim c^t S^{\gamma t}/S^t$. If $\gamma = 0$ and $D \to c$ the numerator is bounded while the denominator diverges. For $t = 0$, $\ln D = \ln c + \gamma \ln S + o(1)$. For $t < 0$, $S^t \to 0$; if $D \to \infty$ also $D^t \to 0$ and $\mathcal{E} \to 1$, whereas $D \to c$ gives $\mathcal{E} \to 1 - c^t$.

C Admissibility under the Chao–Ricotta requirements

Chao and Ricotta (2019) require of an evenness measure: transfer, continuity and invariance to relabeling (1a, 1b); no increase when a species of vanishing abundance is added with negligible change in diversity (2); a fixed limiting range independent of richness (3a); and invariance to

the units of raw abundance (3b). For every $S \geq 2$, $\alpha > 0$ and finite β, $\mathcal{E}_{\alpha,\beta}$ satisfies all of them. At fixed S and β it is strictly increasing in D_α, and D_α is strictly Schur-concave for $\alpha > 0$ (Marshall et al., 2011), which gives the transfer property; continuity and symmetry are inherited. Since $1 \leq D_\alpha \leq S$, the range is $[0, 1]$ with maximum attained only at uniformity (3a). Multiplying all raw abundances by a constant leaves p and hence $\mathcal{E}$ unchanged (3b). For (2), hold $D_\alpha > 1$ fixed and treat S as continuous: $\partial_S \mathcal{E}_\beta = -t\, S^{t-1}(D^t - 1)/(S^t - 1)^2 < 0$ for $\beta \neq 1$ (and $-\ln D/(S \ln^2 S) < 0$ at $\beta = 1$). Adding a category of abundance $\varepsilon \to 0^+$ raises S by one while D_α converges to its earlier value for every $\alpha > 0$, which gives the required non-increase in the limit.

D Two boundary cases

Euclidean Minkowski evenness. For distance order $r = 2$ the Minkowski evenness of Chao and Ricotta (see also Kvålseth, 2015) is $M_2 = 1 - \sqrt{(S - D_2)/(D_2(S - 1))}$, because $\sum_i (p_i - 1/S)^2 = 1/D_2 - 1/S$. It is a function of D_2 alone. As $D_2 \uparrow S$, $1 - M_2 \sim \sqrt{(S - D_2)/(S(S - 1))}$, so $\partial M_2/\partial D_2$ diverges, whereas every finite $\mathcal{E}_\beta$ has finite slope at uniformity, $(1-\beta)S^{-\beta}/(S^{1-\beta}-1)$ (and $1/(S \ln S)$ at $\beta = 1$). Hill-sufficiency at order two therefore does not imply membership in the family.

Mean pairwise similarity. Chisholm's (2026) index $1-G = 1-S^{-1}\sum_{i<j} |p_i - p_j|$ is not a function of any single Hill number. The assemblages $(0.45, 0.30, 0.20, 0.05)$ and $(0.50, 0.20, 0.15, 0.15)$ have the same $\sum p_i^2 = 0.335$ and hence the same $D_2 = 200/67 \approx 2.985$, but $1 - G = 0.675$ and 0.725.

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CRediT authorship contribution statement

Luca Delvecchio: Conceptualization, Methodology, Formal analysis, Investigation, Visualization, Writing – original draft, Writing – review & editing.

Data availability

No original field data were collected. The Alpine illustration uses the public relative-abundance data accompanying Chao and Ricotta (2019), archived at Zenodo (DOI: 10.5281/zenodo.3341384). The Miocene–Pliocene illustration uses coverage-standardized values published in Table 2 of Chao et al. (2020). The supplementary files are `numerical_checks.py` (high-precision numerical checks of the statements above), `evenness_crossing.py` (computes β^* and the verdicts of E_3, E_5, E_4 for two assemblages) and `make_figures.py` (Figures 2–4). All inputs appear in the manuscript.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this manuscript the author used Claude (Anthropic) and ChatGPT (OpenAI) for literature exploration, drafting and language editing, LaTeX preparation, code for numerical checks and figures, and the verification of several mathematical results, including Proposition 3 and Corollaries 1 and 2. The author reviewed and edited the outputs and takes full responsibility for the content of this article.

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