Emergent competition resolves the paradox of stable microbial mutualisms

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Abstract

What is the role of cooperation in determining ecosystem structure? This question is central to ecology, yet remains controversial; cooperative mechanisms such as plant-insect pollination and microbe-microbe cross-feeding are widespread, but ecological theory suggests that cooperative communities should be unstable. Here, we resolve this apparent contradiction by deriving a precise and general mapping between mechanistic interaction models and the generalised Lotka-Volterra (gLV) equation. By avoiding the common assumption that environmental and population dynamics occur on different timescales, we obtain a framework that is built on the rate of change of growth rates, *i.e.* growth accelerations. We apply our findings to a model of an obligatory mutualism between auxotrophic bacteria and obtain a counter-intuitive result: increasing the strength of cooperative mechanisms by increasing amino acid production rates causes competition to become stronger in the corresponding gLV model. At the same time, effective intrinsic growth rates become increasingly positive. These coupled effects allow enhanced cross-feeding to increase overall population sizes while maintaining ecosystem stability. Finally, we provide mechanistic insights into the resulting coexistence outcomes by exploiting an exact correspondence between mechanistic models and Modern Coexistence Theory (MCT) revealed by our framework. Our findings show that cooperative and competitive interaction mechanisms do not necessarily translate into positive and negative interaction strengths in phenomenological models, highlighting the importance of mechanistic insights for developing a correct understanding of ecosystem structure.

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1 Introduction

Mutually-beneficial interactions between populations, or mutualisms, are central to the establishment and persistence of organisms in ecosystems across scales [1]. While originally studied in the context of relationships between macroscopic organisms such as plants and their pollinators or cleaner fish and their clients, mutualistic relationships are now known to be fundamental to the composition of many microbial ecosystems [2]. In the most extreme cases, isolates initially identified as pure cultures of single species have been unmasked as assemblies of mutually-dependent bacteria upon closer examination [3, 4]. More broadly, auxotrophic microbes that lack the ability to synthesise essential vitamins, amino acids and/or neucleotides are widespread in the environment and human-associated microbiomes [5–7]. Such metabolic deficiencies must be compensated for by secretions from surrounding organisms to allow auxotrophic species to persist. The diversity and function of microbiomes is thus strongly influenced by nutritional exchanges [2, 8, 9].

Surprisingly, despite these widespread observations of mutualisms in nature, theoretical investigations based on the generalised Lotka-Volterra (gLV) framework and related models suggest that mutualisms should destabilise ecosystems [10–12]. Reconciling these apparently contradictory results has been a pressing challenge for theoretical ecologists in recent decades [13–17]. While introducing more realistic representations of the underlying biological processes mediating interactions can enhance the stability of mutualisms [18–21] , these modifications remain largely *ad hoc* as they are not derived from an explicit mechanistic representation of the system. Instead, they are based on intuitive but untested assumptions of how changes to interaction mechanisms should impact interaction values.

There is a fundamental technical barrier to resolving this problem using theory: while we can explicitly represent the mechanisms that mediate mutualisms (*e.g.* nutritional exchanges [22–24]) relatively straightforwardly, we have thus far been unable to identify a generic mapping between these mechanistic models and phenomenological interaction frameworks such as the gLV model. Achieving such a mapping is a long sought-after goal in theoretical ecology, first attempted by MacArthur in his pioneering work on resource competition [25]. However, despite decades of efforts to understand the connections between these two frameworks [24, 26–30], a general formulation for identifying equivalencies between the two remains elusive. Several studies have highlighted fundamental differences between their predictions and structures,

suggesting that a direct correspondence may not exist [31–33].

In this work, we resolve this challenge by deriving a general and precise mapping between mechanistic and phenomenological interaction frameworks that is based on growth accelerations rather than rates. Conveniently, our results allow us to directly apply much of the theoretical toolset built to study the gLV equation to mechanistic interaction models. By applying these tools to a two-population model of cross-feeding auxotrophic bacteria, we show that manipulating mechanistic parameters can result in counter-intuitive effects on effective gLV parameters. Specifically, we show that increasing the production rate of cross-fed compounds paradoxically increases effective competition between mutualists. This effect emerges because cross-fed nutrients become less growth limiting as their production is increased. Consequently, growth limitation imposed by non cross-fed resources that are consumed by both mutualists becomes more dominant. Changes in coexistence outcomes resulting from these mechanistic changes can be exactly captured by applying Modern Coexistence Theory (MCT) through our framework, with stronger cross-feeding exchanges increasing niche overlap. Our findings thus provide an unexpected explanation for the stability of microbial mutualisms that is rooted in an understanding of interaction mechanisms.

2 Results

2.1 A generic mapping between mechanistic and phenomenological interaction frameworks

In phenomenological frameworks such as the gLV model, growth rates are shaped purely by the population densities (Fig. 1a). The gLV model treats interaction strength as a linear function of partner density on a focal populations' per-capita growth rate (PCGR, defined as $\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}$):

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = \mu_\alpha + \sum_\beta a_{\alpha\beta}x_\beta. \tag{1}$$

Here, x_α is the instantaneous density of focal population α, μ_α is the 'intrinsic growth rate' of α (its growth rate at low densities and in the absence of other populations) and $a_{\alpha\beta}$ are the interaction strengths (the growth rate impact of one unit of population β on α). These can be assembled into a matrix of values A called the interaction matrix [34]. The equilibrium solution

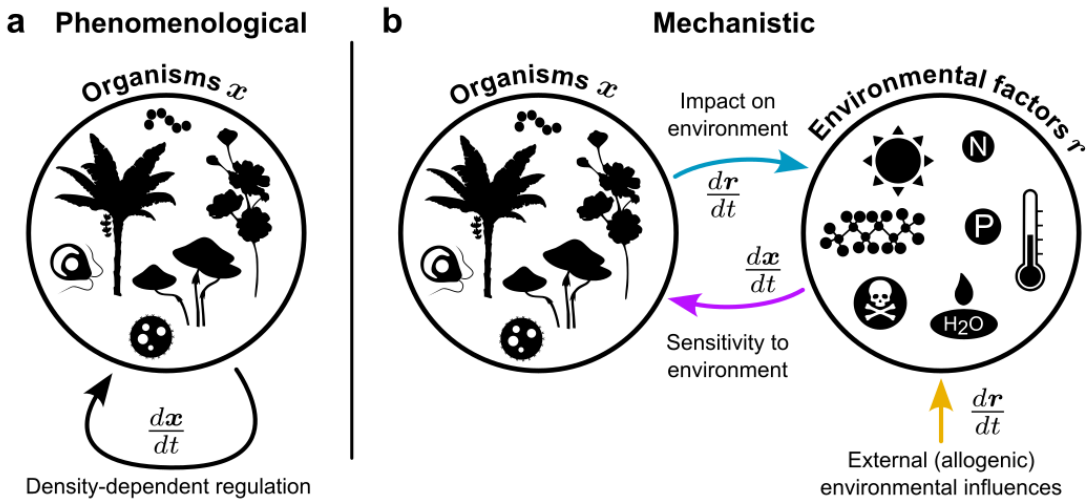

Figure 1: **Structures of phenomenological and mechanistic ecosystem models.** Colours of arrows in panel b correspond to the colour scheme used throughout this manuscript for impact functions (blue), sensitivity functions (purple) and allogenic functions (gold).

to such a system is called 'feasible' if the equilibrium densities (given by $\boldsymbol{x}^* = -\boldsymbol{\mu}A^{-1}$) are strictly positive.

Alternatively, interaction mechanisms can be explicitly represented by splitting the ecosystem into two groups of dynamical quantities: the sizes of the populations of interest $\boldsymbol{x}$ and the quantities of environmental factors $\boldsymbol{r}$ that influence these populations' growth rates (Fig. 1b). Although often treated as the concentrations of abiotic environmental compounds in plant and microbial community ecology [35, 36], these are more general and may also include factors such as biotic resources that can self-replicate [25]. The key structural determinant of these models is that they are bipartite: $\boldsymbol{x}$ can influence $\boldsymbol{r}$ and $\boldsymbol{r}$ can influence $\boldsymbol{x}$, but there are no direct relationships between different members of $\boldsymbol{x}$ or $\boldsymbol{r}$.

The dynamics of the environmental factors are determined by a combination of impacts by the community, which we represent with environment-dependent impact functions $\boldsymbol{f}_\beta(\boldsymbol{r})$ [30], and extrinsic processes that are independent of the community (*e.g.* fluxes of compounds into the ecosystem or reproduction of biotic resources) which we represent with the allogenic function $\boldsymbol{\sigma}(\boldsymbol{r})$. Putting these together, we can then write the rate of change of the environmental factors $\boldsymbol{r}$ as:

$$\frac{d\boldsymbol{r}}{dt} = \sum_\beta x_\beta \boldsymbol{f}_\beta(\boldsymbol{r}) + \boldsymbol{\sigma}(\boldsymbol{r}) \tag{2}$$

The *sensitivity function* g_α describes the growth rate of a population α in a particular environment:

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = g_\alpha(\boldsymbol{r}). \tag{3}$$

While these are often described as the basic equations of consumer-resource models [37], we refer to this class of system more generally as Environment-Organism (EO) models to emphasise that environmental factors need not be resources and that population need not consume them [36, 38]. The impact and sensitivity functions together encode the mechanisms through which the populations x interact, and can be constructed to describe a very broad range of biological processes (Table S1).

Identifying a general way to convert these expressions into a gLV-like equation (Eq. 1) is desirable as it would allow us to understand how underlying interaction mechanisms ultimately shape coexistence. Current techniques for performing this mapping rely on approximations such as the separation of timescales between environmental and population dynamics [24, 25, 30, 39] or low population densities [29], or are applicable only to specially-selected EO systems with convenient algebraic properties [26, 27, 35]. A common feature of most such approaches is their adoption of a first order perspective, aiming to derive an expression that describes the current growth rates in terms of the population densities as in the gLV equation. However, interactions in mechanistic models result from the combination of two rate-based processes [40, 41]: the rate at which organisms impact the environment and the sensitivity of their growth rates to environmental conditions. The combination of two first order processes might be expected to lead to second order dynamics, *i.e.* growth accelerations.

Following this logic, we can evaluate the temporal derivative of the populations' PCGRs to obtain the per-capita growth acceleration (PCGA, $\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right)$):

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \nabla g_\alpha \cdot \frac{d\boldsymbol{r}}{dt}, \tag{4}$$

where we have used the multivariable chain rule to split the temporal derivative of the sensitivity function into the sensitivity gradients with respect to the environmental factors ($\nabla g_\alpha \equiv \frac{\partial g_\alpha}{\partial \boldsymbol{r}}$) and the rate of environmental change. Substituting in Eq. 2, we obtain (Supplementary Note

2.1):

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \mu'_\alpha(\boldsymbol{r}) + \sum_\beta a'_{\alpha\beta}(\boldsymbol{r})x_\beta. \tag{5}$$

This equation, which we will refer to as the accelerational Environment-Organism (aEO) equation, partitions the PCGA into two parts: the abiotic acceleration $\mu'_\alpha(\boldsymbol{r}) \equiv \boldsymbol{\sigma}(\boldsymbol{r}) \cdot \nabla g_\alpha(\boldsymbol{r})$ represents the rate of change of the growth rate of α due to the extrinsic drivers of the environmental factors, while the term $\sum_\beta a'_{\alpha\beta}(\boldsymbol{r})x_\beta$ captures the growth rate impacts caused by the community's impact on the environmental factors. Central to this term are the biotic accelerations $a'_{\alpha\beta}(\boldsymbol{r}) \equiv \boldsymbol{f}_\beta(\boldsymbol{r}) \cdot \nabla g_\alpha(\boldsymbol{r})$, which capture the per-capita contribution to this effect from each community member.

Despite the fact that the gLV equation is first order and the aEO equation is second order, there are deep equivalencies between the two. Notably, at an equilibrium state in environment $\boldsymbol{r}^*$, $\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = 0$. We therefore obtain

$$0 = \mu'_\alpha(\boldsymbol{r}^*) + \sum_\beta a'_{\alpha\beta}(\boldsymbol{r}^*)x_\beta, \tag{6}$$

i.e. an expression that is algebraically identical to the gLV equation (Eq. 1) at equilibrium. This result means that for many rate-based mathematical statements which are true for an equilibrium gLV system, there is a parallel acceleration-based statement which holds for all equilibrium EO systems. We discuss some of the tools that can be built through this parallelism in Supplementary Note 2.3. This result is exact and general, only requiring that $g_\alpha(\boldsymbol{r})$ is differentiable with respect to the environmental factors, such that the sensitivity gradient $\nabla g_\alpha(\boldsymbol{r})$ is well-defined.

2.2 A simulated cross-feeding community demonstrates counter-intuitive relationships between interaction mechanisms and effective gLV parameters

To illustrate how these results inform our understanding of mutualistic exchanges in microbial ecosystems, we built a two-population EO model representing obligate cross-feeding between two auxotrophic bacterial populations, $\mathring{v}$ and $\mathring{w}$, which require external supplies of amino acids

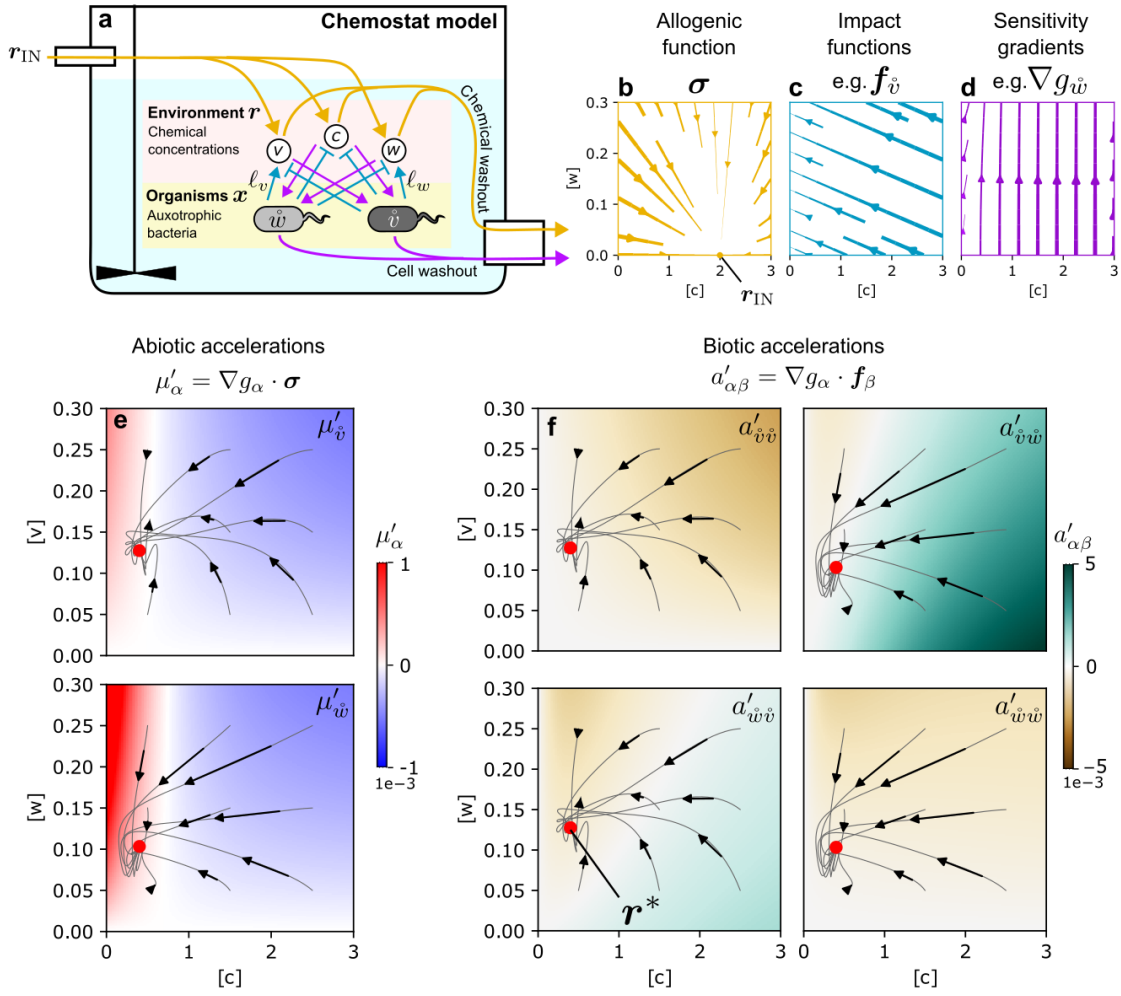

Figure 2: **An accelerational description of cross-feeding auxotrophs at equilibrium.** EO models consist of three components – allogenic functions, sensitivities and impacts - that together represent the mechanisms through which organisms interact. a) We built an EO model representing a community of two cross-feeding auxotrophic bacteria (each unable to synthesise one of two amino acids v and w) in a chemostat. Metabolic dependencies on a carbon source c and amino acids are encoded by sensitivity functions (purple), while consumption and production are represented by impact functions (blue, arrows indicate production, bars consumption). These functions are constructed using explicit mechanistic parameters such as the amino acid production rates ℓ_v and ℓ_w. Allogenic processes in this system (gold) are the external influences on chemical concentrations, *i.e.* nutrient influx and washout of chemicals through the chemostat efflux. Washout of cells at the same rate also acts as an effective mortality term. b-d) These three processes can be represented as vector fields in the 'environment space', the space of possible environmental compositions $\boldsymbol{r}$. e,f) Taking the scalar product of the sensitivity gradients and allogenic processes yields the abiotic accelerations (e), while the scalar product of the sensitivity gradients and impact functions yields the biotic accelerations (f). These can be represented as scalar fields sitting in the environment space. The chemical environment converges on an equilibrium composition $\boldsymbol{r}^*$ (red circle) regardless of the starting composition of the media in the chemostat (grey trajectories and arrows). Note that the full environment space is three dimensional, representing the concentrations of c, v and w (Fig. S2). In b-d, we display slices through this space for which the concentration of the chemical corresponding to the suppressed axis is set to zero. Environmental trajectories (c,d) are projected onto this plane. Arrow thickness in b indicates the local magnitude of the vector field. Dilution rate $D = 0.01$.

v and w respectively. This type of system is a popular choice for studying mutualisms in the lab [23, 42–46] and is widely observed in environmental microbiomes [2, 4, 6]. Our formulation (Methods) assumes that growth depends on co-utilisation of a carbon source c (used by both partners) and a population-specific amino acid (v or w) that is secreted by the mutualistic partner. We also include 'overflow' production of amino acids when an auxotroph's essential amino acid is at low concentrations but the concentration of the carbon source is high, reflecting recent experimental findings [47] (Fig. 2a). This community was then placed into a simulated chemostat, an idealised representation of systems with continual inflows and outflows of material such as the gut [48]. In the absence of external amnio acids, this model has multiple features of obligatory mutualisms including bistability and Allee effects [20] (Fig. S1).

The mechanistic processes that define an EO model can be represented as vector fields sitting in the 'environment space', which represents all possible combinations of the environmental factors (Figs. 2b-d, S2). The allogenic function and impact functions indicate the direction that the environment is pulled by the abiotic and biotic influences on the environment, respectively. For example, dilution of a chemostat's content with media of composition $\boldsymbol{r}_{\text{IN}}$ results in an allogenic function in which all vectors point towards $\boldsymbol{r}_{\text{IN}}$ (Fig. 2b). Sensitivity gradients can also be represented as vector fields, now representing the rate at which a population's growth rate changes as the environment is pulled in a given direction (Fig. 2d). Taking scalar products of these vector fields give scalar fields that represent the rate at which the growth rate of population α is modified by the external drivers of the environmental dynamics (the abiotic acceleration, $\mu'_\alpha(\boldsymbol{r})$) or by the effect of another community member β on the environment (the biotic accelerations, $a'_{\alpha\beta}(\boldsymbol{r})$.

While $\mu'_\alpha(\boldsymbol{r})$ and $a'_{\alpha\beta}(\boldsymbol{r})$ can change as environmental conditions fluctuate [36] (Fig. S3), we will generally focus on their values in an equilibrium environment $\boldsymbol{r}^*$, *i.e.* $a'_{\alpha\beta}(\boldsymbol{r}^*)$ and $\mu'_\alpha(\boldsymbol{r}^*)$ (Fig. 2c,d). Under these conditions, they can be treated as effective gLV parameters through the algebraic correspondence between Eqs. 6 and 1. Beyond this point, we will suppress the explicit dependency on $\boldsymbol{r}^*$ for notational compactness.

In previous studies, the stability of such two-population mutualisms has been investigated through modified gLV equations that introduce new quantities – for example, terms that cause facilitation to saturate or become competitive as the donor's density increases [21, 49]. These modifying terms are, in effect, intended to impose non-linear density-dependencies on top of

the linear density-dependencies of the Lotka-Volterra model that more realistically represent the mechanisms underpinning the mutualism. However, as interaction mechanisms are not explicitly represented in such models, the proposed relationship between mechanisms and effective gLV parameters remains a choice that must be intuited by the modeller. By contrast, the mechanistically explicit mapping between Eqs. 1 and 6 allows us to avoid this intuitive leap. How does this affect our understanding of the stability of mutualisms?

To investigate this question, we asked how the biotic accelerations $a'_{\alpha\beta}$ and abiotic accelerations μ'_α (which are algebraically homologous to interaction strengths $a_{\alpha\beta}$ and intrinsic growth rates μ_α respectively) vary as the amino acid production rate ℓ_w is increased, representing an evolutionary scenario in which one population unilaterally modifies its cross-feeding strategy [50]. From a naïve interpretation of the gLV model, we would expect this to result in stronger positive interactions from the producer to the consumer, while the costs associated with amino acid production would drive increasingly negative intrinsic growth rates for the producer [10, 20].

To test whether this prediction held in our mechanistic model, we allowed the simulated chemostat to reach an equilibrium for each value of ℓ_w and recorded the equilibrium densities of $\mathring{v}$ and $\mathring{w}$ (Fig. S4a, lines), biotic accelerations $a'_{\mathring{v}\mathring{v}}$, $a'_{\mathring{v}\mathring{w}}$, $a'_{\mathring{w}\mathring{w}}$ and $a'_{\mathring{w}\mathring{v}}$ (Fig. 3a) and abiotic accelerations $\mu'_{\mathring{v}}$ and $\mu'_{\mathring{w}}$ (Fig. 3b). Surprisingly, while the biotic accelerations were initially positive, they switched signs to negative as the strength of the cooperative mechanism was increased. By contrast, the abiotic accelerations grew in value, becoming increasingly positive after being negative at low production rates. While these relationships were unexpected, the accelerational analogue of the expression relating equilibrium densities to gLV parameters ($\boldsymbol{x}^* = -\boldsymbol{\mu}' A'^{-1}$) accurately yielded the equilibrium densities $\boldsymbol{x}^*$ (Fig. S4a, circles), confirming the algebraic correspondence between Eqs. 1 and 6. We also found that the real part of the system's eigenvalue spectrum was negative for all production values (Supplementary Note 2.3.2, Fig. S5), indicating linear stability of coexistence (Fig. S4b). These results are not an artefact of the small size of the community or the choice of mechanistic parameters, as they generalise to larger communities with random parameters [51] (Fig. S6). This suggests that our intuitive understanding of the relationship between interaction mechanisms and effective gLV parameters is flawed: increased cross-feeding leads to stronger mutual growth but also stronger competition.

To gain a deeper understanding of the processes driving this paradoxical relationship, we turned

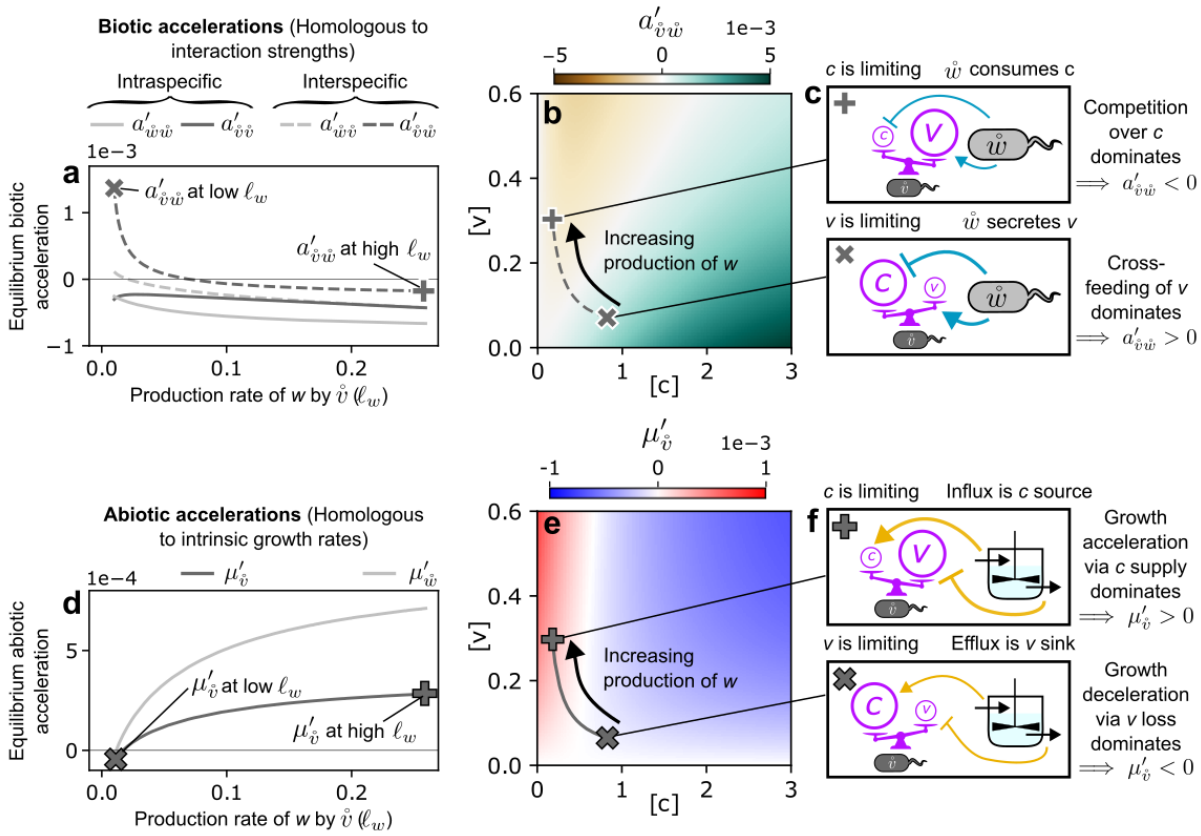

Figure 3: **Modulation of auxotroph amino acid production rates causes counter-intuitive changes to effective gLV parameters.** a,d) Keeping all other parameters fixed, we varied the rate at which $\mathring{v}$ converts the carbon source to secreted w, ℓ_w, and numerically simulated each chemostat's dynamics until it reached an equilibrium. We then evaluated the equilibrium biotic accelerations for each pair of populations (a) and abiotic accelerations for each population (d). b,e) Counter-intuitive relationships between the varying strength of the cooperative cross-feeding mechanism and the effective gLV parameters can be understood by plotting the changing position of $\boldsymbol{r}^*$ on the scalar fields representing one of the biotic (b) and abiotic (e) accelerations (replotted from Fig. 2 e and f). c,f) Schematics representing the mechanisms driving changes in effective gLV parameters under different production rates of w. $\times$ and $+$ symbols represent simulations with the lowest and highest production rates of w, respectively. Dilution rate $D = 0.01$.

to the scalar fields representing the inter-specific biotic acceleration $a'_{\mathring{v}\mathring{w}}$ (Fig. 3b) and the abiotic acceleration $\mu'_{\mathring{v}}$ (Fig. 3e). We selected these over $a'_{\mathring{w}\mathring{v}}$ and $\mu'_{\mathring{w}}$ as they are independent of ℓ_w and thus remain fixed over the parameter sweep. As the production of w increases, the equilibrium concentration of c decreases while the concentration of v increases, driven by the increased density of the w auxotroph. In the low production scenario, the concentration of the amino acid is growth limiting and cross-feeding of v from $\mathring{w}$ dominates over competition for carbon, leading to a positive net value of $a'_{\mathring{v}\mathring{w}}$. By contrast, carbon limitation in the high production scenario leads to dominance of carbon-based competition between the two strains, creating a negative net value (Fig. 3c). A similar argument applies to the abiotic accelerations, but with the effect of the external processes (*i.e.* the fluxes of chemicals into and out of the chemostat) substituted in place of the mutualistic partner; domination of loss of amino acids through the system's efflux drives a negative net value of $\mu'_{\mathring{v}}$ in the amino-acid limited low production scenario, while the positive growth rate impact of the carbon supply dominates in the carbon-limited high production scenario (Fig. 3f).

2.3 How nutrient exchanges shape coexistence: mapping mechanisms to MCT

With a clearer understanding of the relationship between mutualistic mechanisms and effective gLV parameters, we are now equipped to move one level higher and understand how mechanisms shape coexistence. Two-population ecosystems have been extensively studied through the lens of Modern Coexistence Theory (MCT), which seeks to partition processes that impact coexistence outcomes into stabilising components that reduce the niche overlap between populations and equalising components that balance the fitness ratio between populations [28]. A classic result arising from two-population Lotka-Volterra models is that explicit expressions for the niche overlap ρ and fitness ratio $\frac{k_1}{k_2}$ can be obtained from the gLV parameters (Supplementary Note 2.3.3) [52, 53]:

$$\rho = \sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}} \tag{7a}$$

$$\frac{k_1}{k_2} = \frac{\mu_1}{\mu_2}\sqrt{\frac{a_{22}a_{21}}{a_{11}a_{12}}}. \tag{7b}$$

where 1 and 2 are labels for the two populations. Coexistence is feasible only if

$$\rho < \frac{k_1}{k_2} < \frac{1}{\rho}. \tag{8}$$

We can define an accelerational analogue of this expression. Using the labels of the auxotrophs, we obtain:

$$\rho' = \sqrt{\frac{a'_{\mathring{v}\mathring{w}} a'_{\mathring{w}\mathring{v}}}{a'_{\mathring{w}\mathring{w}} a'_{\mathring{v}\mathring{v}}}} \tag{9a}$$

$$\frac{k'_{\mathring{w}}}{k'_{\mathring{v}}} = \frac{\mu'_{\mathring{w}}}{\mu'_{\mathring{v}}} \sqrt{\frac{a'_{\mathring{v}\mathring{v}} a'_{\mathring{v}\mathring{w}}}{a'_{\mathring{w}\mathring{w}} a'_{\mathring{w}\mathring{v}}}}. \tag{9b}$$

Mutualisms have previously been thought to be incompatible with this form of MCT, as positive intraspecific interaction strengths lead to imaginary niche overlaps [54]. However, we have already shown that values of $a'_{\alpha\beta}$ are negative for most amino acid production rates, suggesting that we may be able to use this framework to analyse many instances of the cross-feeding system.

To test whether we could use this approach to shed light on the mechanistic basis of coexistence in this system, we manipulated the amino acids through two mechanisms: varying their concentrations in the input media r_{IN} (Fig. 4a-c) and varying their production rates by the two populations (Fig. 4d-f). We then scored communities based on whether one, both or neither population was able to persist at equilibrium (Fig. 4b,e). Finally, we mapped these scenarios onto niche overlap and fitness ratio axes using Eq. 9b to verify the accuracy of the coexistence predictions by MCT (Fig. 4c,f). Some communities displayed positive biotic accelerations, which were excluded from the niche overlap and fitness ratio calculations to avoid imaginary values. These tended to be located at the edges of the coexistence regions where amino acid concentrations were particularly limiting. For the remaining communities, we found that Eq. 9b precisely predicts the coexistence properties of this system, with combinations of mechanistic parameters that result in competitive exclusion mapped to the boundaries of the coexistence region.

How do coexistence outcomes change as underlying mechanisms are varied? Changes in the ratio of the input amino acid concentrations $[v]_{\text{IN}}$:$[w]_{\text{IN}}$ (Fig. 4b,c, dashed line with black arrow-

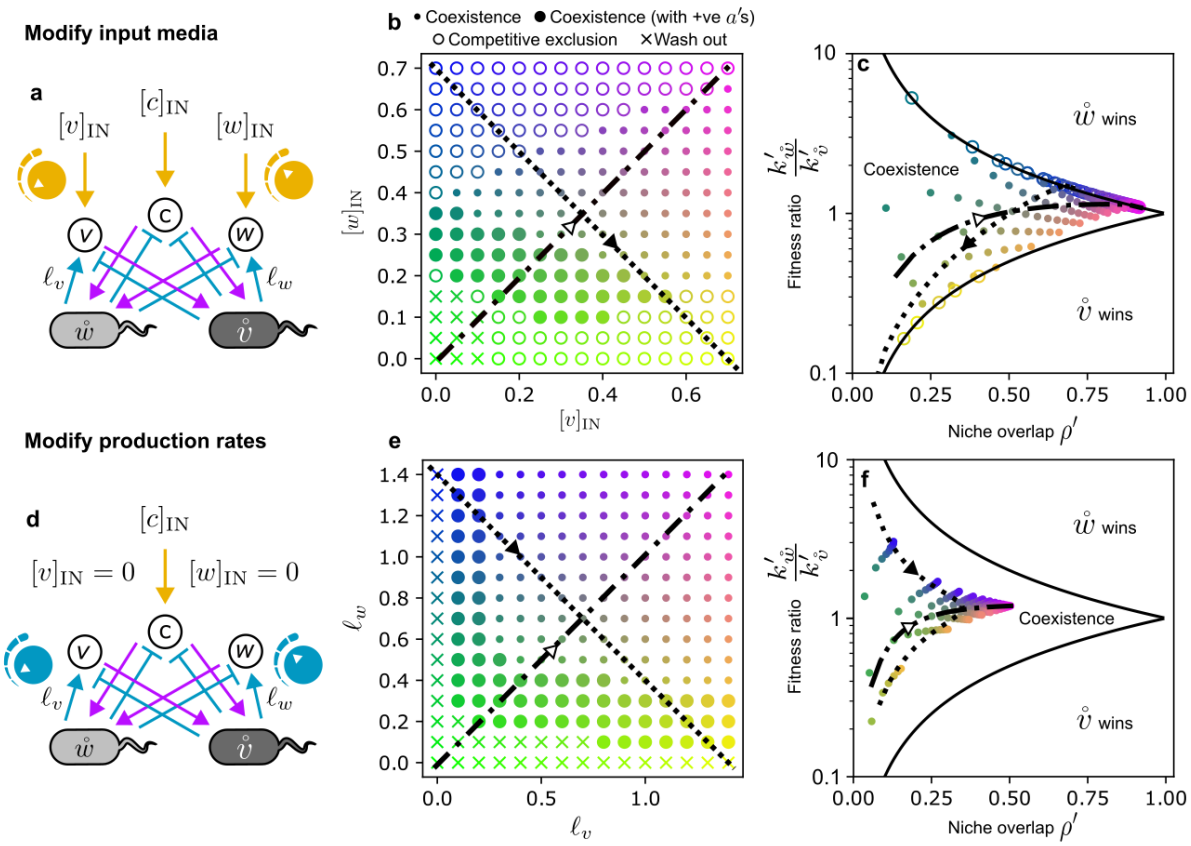

Figure 4: **Mapping interaction mechanisms onto modern coexistence theory** a) To test the validity of our accelerational definition of the niche overlap and fitness ratio, we first varied the input concentrations of the two amino acids in the two population auxotrophy model. b) We then performed numerical simulations of the population dynamics in the chemostat for each combination of input concentrations, scoring coexistence outcomes based on whether both populations were washed out (crosses), one population competitively excluded the other (open circles) or both populations stably coexisted (filled circles). c) The abiotic and biotic accelerations for both strains in the equilibrium environment $\boldsymbol{r}^*$ were then converted to niche overlaps ρ' and fitness ratios $\frac{k'_{\mathring{w}}}{k'_{\mathring{v}}}$ (main text). Solid lines indicate predicted coexistence boundaries (Eq. 8). Note that the contours of constant input amino acid ratio (dot-dash line) and total input amino acid concentration (dotted line) are mapped onto the niche overlap and fitness ratio axes. d-f) Equivalent analysis conducted with varying amino acid production rates from the two auxotrophs. Large filled circles in b and e denote stably coexisting communities with at least one positive biotic acceleration a', which are excluded from the niche overlap and fitness ratio axes in c and f. Simulations resulting in mutual wash out are also excluded from these axes. Points with matching colours in b, c and e, f correspond to the same simulation in the two panels. Dilution rate $D = 0.03$, input media contains no amino acids in d-f.

head) predominantly drove changes in fitness ratios, with the less supported population being outcompeted as soon as the system hit a coexistence boundary. On the other hand, increasing the total amino acid concentration while keeping their ratio fixed (Fig. 4b,c, dot-dashed line with white arrowhead) mainly changed the niche overlap, leading to a progression from mutual wash-out to coexistence and finally competitive exclusion as observed in experiments [23]. A similar analysis on amino acid production rates (Fig. 4e,f) yielded closely related patterns. However, we found that no combination of production rates resulted in competitive exclusion, reflecting their mutual dependence in media lacking amino acids. Instead, correlated changes in the niche overlap and fitness ratio prevented the system from hitting the coexistence boundary (Fig. 4f).

We now investigated the mechanistic basis for these patterns. Adopting a sampling-based approach, we generated communities with randomised amino acid production rates and input concentrations and scored the fitness ratio and niche overlap of each pair of strains at equilibrium. We then binned communities according to their values of these two quantities and calculated the average nutrient concentrations, mechanistic parameters (Fig. 5a-g) and effective gLV parameters (Fig. S7) for all systems falling within these bins. This allowed us to relate variations in the MCT metrics to underlying mechanistic processes (Fig. 5f): at low amino acid concentrations, the two populations are not competing for the carbon source as their growth is limited by different amino acids. Processes that increase amino acid concentrations relieve this non-competitive growth constraint and instead shift the populations towards competition for the carbon source on which they are mutually dependent, driving an increase in niche overlap (Fig. 5a-c). Fitness ratios on the other hand are determined by the relative fluxes of the two amino acids, with imbalanced fluxes favouring the auxotroph receiving the stronger supply of their limiting nutrient (Fig. 5d-g).

This perspective also allows us to explain why the system responds differently when the imbalance in fluxes arises due to internal (production rates) versus external (input concentrations) changes in mechanisms. Similar to Song *et al.* [55], we find that mechanistic changes are not purely stabilising nor equalising. Introducing an imbalance in production rates leads to an imbalance in the fitness ratio. However, it also leads to weaker overall growth due to the mutual dependence of the populations. This in turn leads to lower niche overlap due to less competition for carbon, allowing coexistence to be maintained (dotted line in Fig. 4f). Similar patterns have

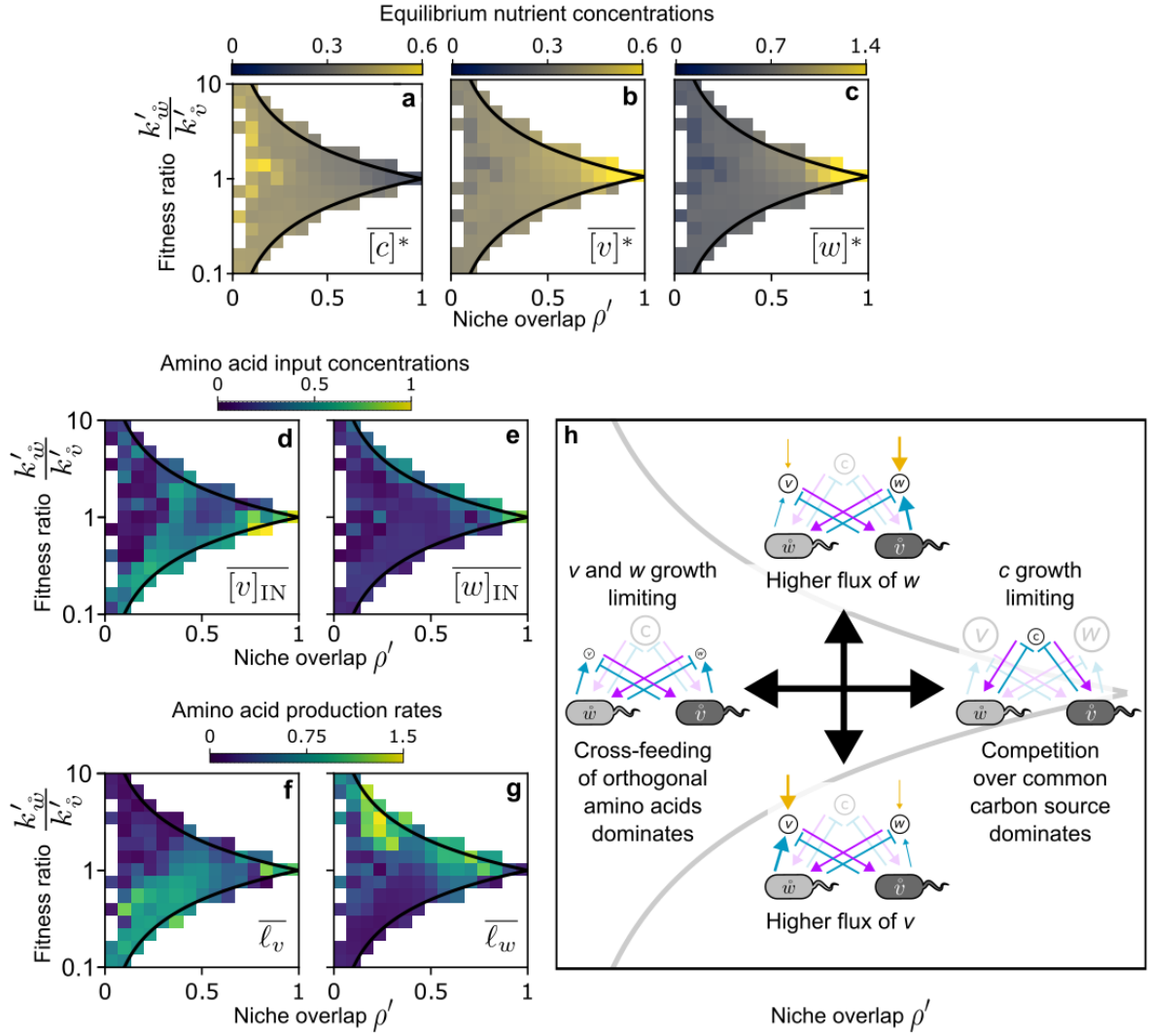

Figure 5: **The mechanistic basis of fitness ratios and niche overlaps of auxotrophic bacteria.** a-g) We generated communities with randomised combinations of amino acid production rates and amino acid input concentrations and evaluated the niche overlap ρ' and fitness ratio $\frac{k'_{\hat{v}}}{k'_{\hat{w}}}$ for each community at equilibrium. We then binned communities along these axes and calculated the average equilibrium carbon source concentration $\overline{[c]^*}$ (a) and amino acid concentrations $\overline{[v]^*}$, $\overline{[w]^*}$ (b,c), as well as the amino acid input concentrations $[w]_{\text{IN}}$, $[w]_{\text{IN}}$ (d,e) and production rates $\overline{\ell_v}$, $\overline{\ell_w}$ (f,g) for all communities falling within each bin. h) Summary of mechanisms driving changes in niche overlap and fitness ratios in the auxotrophy model. As in Fig. 4, communities with positive biotic accelerations were excluded from the analysis. Dilution rate $D = 0.03$.

been observed experimentally, with increased production of an amino acid by one member of a cross-feeding mutualism favouring the recipient but never leading to competitive exclusion [46]. On the other hand, imbalances in externally supplied amino acid concentrations are not subject to this internal regulation through changes in niche overlap. This allows imbalances in the fitness ratio to drive the system into the coexistence boundary, enabling competitive exclusion (dotted line in Fig. 4c).

3 Discussion

The relative roles of cooperation versus competition in ecosystems is a long-debated topic in ecology [14–16, 56–58]. Within microbiology, this controversy is perhaps best reflected in the apparently contradictory results of studies that measure interaction strengths, which are predominantly competitive [59], versus those that study interaction mechanisms, which find widespread mutualistic exchanges [2]. Our results reconcile these two camps by showing that the term 'mutualism' means two different things, depending on the ecosystem property being investigated. The auxotrophy system that we model here is a paradigmatic obligate mutualism in terms of the ability of the two strains to persist (Fig. S1). Yet when we look at the quantities algebraically equivalent to the interaction strengths of the gLV model $a_{\alpha\beta}$, the biotic accelerations $a'_{\alpha\beta}$, we observe that competition dominates and becomes stronger as cross-feeding is strengthened (Fig. 3). This is generally expected to stabilise communities [11, 12]. Cross-feeding communities can thus be mutualistic in terms of their persistence properties, but competitive as far as their stability is concerned.

In line with these results, previous studies have shown that the restraining effect of competition over shared resources can prevent mutualisms from destabilising ecosystems. Aside from carbon availability, these limiting factors can include nitrogen, space and the density of shared partners [11, 33, 45, 60]. Similar arguments have also been made for groups of macroscopic organisms, such as complexes of Müllerian mimics that compete for nectar [61]. However, an important outstanding question in the context of gLV-type models is why this enhanced competition does not also suppress population sizes. Our results show that increased competition is compensated for by a paradoxical increase in the term that plays the role of the intrinsic growth rates μ_α, the abiotic accelerations μ'_α (Fig. 3). This finding contrasts with the typical assumption that μ_α is either fixed or declines with increasing costs of cooperation, demonstrating that

though algebraically equivalent at equilibrium, the gLV and aEO equations have fundamentally different interpretations. Indeed, there is nothing 'intrinsic' about μ'_α, which is largely determined by the extrinsic environmental inputs. Studies which assume that intuitive relationships hold between interaction mechanisms and effective gLV parameters should be revisited in light of this alternative interpretation.

These findings also impact our understanding of the niche overlap and fitness ratio metrics of MCT. While our results are consistent with previous geometric interpretations of niche overlap as the degree of alignment between requirements or impacts of populations [27, 29, 52], our interpretation of the fitness ratio is more difficult to reconcile with existing work; certainly it is not obvious how to interpret the observed relationships as variations in fitnesses. While our findings suggest that it represents the extent to which fluxes of environmental quantities favour a particular population (Fig. 5h), additional EO models should be interrogated to assess the universality of this interpretation. Regardless of their interpretation however, the ability to map mechanistic processes onto the metrics of MCT represents a powerful simplification of ecosystems, allowing us to distil ecosystem processes involving an almost unlimited number of environmental factors down to two numbers that completely determine coexistence [62].

The main technical advance of our work is the derivation of a general and exact method for transforming mechanistic ecosystem models into a gLV-like parametrisation, allowing tools used to analyse gLV systems to be applied to mechanistic models. MacArthur's solution to this problem, based on the assumption that resources equilibrate much more rapidly than populations [25], has dominated the literature since its discovery [52, 55, 63–65]. However, this technique is approximate and can only be applied in cases where the dynamics of resources are logistic [26], a serious limitation when modelling the dynamics of ecosystems based on abiotic nutrients such as microbiota. Our approach avoids eliminating this timescale of the resource dynamics, consequently resulting in a second-order description of the population dynamics. It remains to be seen whether this accelerational picture proves a more accurate description of real ecosystems than existing rate-based representations, although we note that changes in population growth rates (*i.e.* growth accelerations) in closed microcosms are accurately described by a limiting case of the aEO equation [36]. Second-order descriptions also enable replication of phenomena not observed in standard first-order systems, such as density oscillations of single populations and time lags in population responses [66–68].

Our results also suggest new solutions to practical problems in microbial ecology. Currently, extensive measurements of environmental quantities are required to couple mechanistic models to experiments, significantly limiting their utility [69]. On the other hand, the gLV model is unable to capture ecosystems governed by indirect interaction mechanisms, and must be made substantially more complex through terms representing higher order interactions (HOIs) [32]. Remarkably, Eq. 6 resolves both of these problems: while it takes the environmentally-mediated interaction structure of microbial communities into account, it contains only single-species and pairwise terms which can be determined without explicit environmental measurements. By combining the advantages of the mechanistic and phenomenological viewpoints, this insight may provide new approaches to the characterisation and control of microbial ecosystems.

However, our approach also has its drawbacks. One of the main attractions of the gLV model and related frameworks are their analytical tractability, forming the basis of a large body of results based on random matrix theory [11, 12, 70]. These results describe the properties of very large communities provided that the associated parameters follow prescribed statistical patterns. By contrast, our approach relies heavily on numerical integration schemes to solve system dynamics and obtain equilibrial environments $\boldsymbol{r}^*$ for which Eq. 6 holds. While techniques exist for calculating $\boldsymbol{r}^*$ analytically for specific models (Supplementary Note 2.2), these generalise poorly and typically require explicit evaluation of matrix inverses. They also assume *a priori* that such a point exists and is an attractor, rather than the system dynamics converging on indefinite oscillations or chaos. Application of random matrix theory through this framework is therefore likely to be challenging.

In summary, our results provide a highly flexible link between the molecular mechanisms underpinning interactions and emergent coexistence outcomes. Application of this framework to the question of how interaction mechanisms constrain ecosystems promises to resolve a number of outstanding problems in ecology beyond the paradox that we address here. Successfully addressing these problems will provide us with new tools for understanding and manipulating ecosystems across scales, including microbiomes.

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Emergent competition resolves the paradox of stable microbial mutualisms – Supplementary Information

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1 Supplementary methods

1.1 Auxotrophy model

1.1.1 Formulation

Species in the auxotrophy model depend on both an in-common carbon source that all members of the community rely on (*e.g.* glucose) and a species-specific amino acid that they alone rely on [1]. The co-essentiality of these two resources is represented by using the product of separate Monod functions for the two resources when determining growth rates and resource utilisation rates. We additionally include a metabolic shunt that enables production of cross-fed amino acids even when the essential amino acid for a given species is not available [2]. This ensures that amino acid secretion is not directly coupled to growth, allowing the community to begin secreting amino acids into the input media even when it contains only the raw carbon source. The community can thus establish in amino-acid free media (Fig. S1). The rate of amino acid secretion is determined purely by the carbon source availability, again transformed via a Monod function.

We arrange the environment vector $\boldsymbol{r}$ so that amino acid in position ρ corresponds to the amino acid required by the species with the same index (*i.e.* so that $\rho = \alpha$ for auxotroph α and required amino acid ρ). The in-common carbon source is placed in the final position of the vector with index $N + 1$ (where N is the number of auxotrophic populations). The sensitivity function is then defined as:

$$g_\alpha = \nu_\alpha \frac{r_\alpha}{r_\alpha + K_{\alpha\alpha}} \frac{r_{N+1}}{r_{N+1} + K_{\alpha(N+1)}} - D, \tag{S.1}$$

where D is the dilution rate of the chemostat, $K_{\alpha(N+1)}$ and $K_{\alpha\alpha}$ are the half-velocity constants for the carbon source and consumed amino acid respectively and ν_α is the maximal growth rate.

The impact function is additionally defined as:

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$$f_{\alpha\rho} = \begin{cases} -\frac{1}{Y_{\alpha\rho}}\nu_\alpha \frac{r_\rho}{r_\rho + K_{\alpha\rho}} \frac{r_{(N+1)}}{r_{(N+1)} + K_{\alpha(N+1)}} & \text{if } \rho = \alpha \text{ } (\rho \text{ is missing amino acid}) \\ -\frac{1}{Y_{\alpha\rho}}\nu_\alpha \frac{r_\alpha}{r_\alpha + K_{\alpha\alpha}} \frac{r_\rho}{r_\rho + K_{\alpha\rho}} - \nu_\alpha \frac{r_\rho}{r_\rho + K_{\alpha\rho}} \sum_{\gamma \neq \{\alpha, N+1\}} \ell_{\alpha\gamma} & \text{if } \rho = N+1 \text{ } (\rho \text{ is carbon source}) \\ \nu_\alpha \ell_{\alpha\rho} \frac{r_{(N+1)}}{r_{(N+1)} + K_{\alpha(N+1)}} & \text{otherwise } (\rho \text{ is secreted amino acid}) \end{cases} \tag{S.2}$$

where we have introduced the scaled production rate of amnio acid ρ by species α as $\ell_{\alpha\rho}$ and the yield of species α on amino acid/carbon source ρ as $Y_{\alpha\rho}$.

Finally, we define the allogenic function using the standard form for chemostat-type systems, $\boldsymbol{\sigma} = D(\boldsymbol{r}_{\text{IN}} - \boldsymbol{r})$, where $\boldsymbol{r}_{\text{IN}}$ is the composition of the input media.

1.1.2 Simulations

We provide parameter values for the 2-species model in table S2. In some cases, we suppress the species label where this is clear from context to simplify notation (for example, $\mathring{v}$ is the only species that secretes w, so we write ℓ_w rather than $\ell_{\mathring{v}w}$). Note that some parameters (ℓ_v, ℓ_w, D, $[v]_{\text{IN}}$ and $[w]_{\text{IN}}$) are modified during some analyses, in which case they are specified on plot axes, in the relevant figure's caption, or were drawn randomly (see next section).

SciPy's `solve_ivp` function was used to numerically integrate the resulting systems of coupled ODEs using the Runge-Kutta method of order 5(4).

1.1.3 Random sampling

Parameters for the randomly sampled mechanistic sweep (Figs. 5, S7) were selected from uniform distributions. The amino acid production rates ℓ_v and ℓ_w were drawn from the range $[0, 1.5]$, while the input concentrations v_{IN} and w_{IN} were drawn from the range $[0, 1]$. The dynamics of the resulting model was numerically integrated until reaching an equilibrium state. The resulting system was then mapped onto niche overlap and fitness difference axes (Supplementary Note 2.3.3) and placed into a 15×15 grid covering the niche overlaps between 0 and 1 and the fitness differences between 0.1 and 10. Mechanistic parameters (Fig. 5 and effective gLV parameters (Fig. S7) were binned according to the position of the final system in this grid. Systems that fell outside this region (including systems with positive values of $a'_{\alpha\beta}$, see main text) were excluded from further analysis. This procedure was repeated 100,000 times, building up a collection of underlying parameters for each bin. The collection in each bin was then averaged to give the final heatmaps.

For the randomly parameterised 5-species model presented in Fig. S6, we drew the values of $K_{\alpha\rho}$, $Y_{\alpha\rho}$, ν_α and $\ell_{\alpha\rho}$ from random distributions. $K_{\alpha\rho}$s were drawn uniformly from the range $[0.75, 1.25]$. $Y_{\alpha\alpha}$s (the yield of the amino acids) were drawn uniformly from the range $[15, 25]$, while $Y_{\alpha,(N+1)}$ (the yield of the carbon source) was set at 1. ν_αs were drawn from $[9, 11]$. $\phi_{\alpha\rho}$s was constructed by drawing values uniformly from the range $[-0.1, 0.4]$ and clipping the negative part of the resulting distribution to zero, simulating amino acids that are not produced by α. We additionally set $\phi_{\alpha\alpha} = 0$ to prevent secretion of the amino acid α was auxotrophic for and $\phi_{\alpha(N+1)} = 0$ to prevent secretion of the carbon source. The resulting set of random values was scaled by a constant that allowed us to vary the average production rate, which was swept systematically from 0.001 to 0.3. The 'average production rate' is the average of all production rates following this scaling. We fed the simulated community in the chemostat on media containing the carbon source at concentration $r_{(N+1)} = 1$ and no amino acids. All populations were inoculated at density 0.05. The dilution rate D was 0.05.

2 Supplementary Notes

2.1 Supplementary Note 1: Derivation of EO equations

To derive the accelerational EO (aEO) equation (Eq. 5), we find the temporal derivative of the PCGR of a focal species α in an EO system. Starting with our definition of the sensitivity function $g_\alpha(\boldsymbol{r})$ as the PCGR at $\boldsymbol{r}$ (Eq. 3), we obtain

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \frac{dg_\alpha(\boldsymbol{r})}{dt} \tag{S.3}$$

Applying the chain rule yields

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \nabla g_\alpha \cdot \frac{d\boldsymbol{r}}{dt}. \tag{S.4}$$

Next, we substitute the definition of the rate of environmental change (Eq. 2)

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \nabla g_\alpha \cdot \boldsymbol{\sigma} + \sum_\beta \nabla g_\alpha \cdot \boldsymbol{f}_\beta x_\beta. \tag{S.5}$$

Applying the definitions of the abiotic acceleration $\mu'_\alpha(\boldsymbol{r}) \equiv \boldsymbol{\sigma}(\boldsymbol{r}) \cdot \nabla g_\alpha(\boldsymbol{r})$ and the biotic acceleration $a'_{\alpha\beta}(\boldsymbol{r}) \equiv \nabla g_\alpha \cdot \boldsymbol{f}_\beta$ yields the aEO equation:

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \mu'_\alpha(\boldsymbol{r}) + \sum_\beta a'_{\alpha\beta}(\boldsymbol{r})x_\beta. \tag{S.6}$$

As noted in the main text, in an equilibrium environment $\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = 0$. Applying this assumption to the aEO equation results in the equilibrium Environment Organism (eEO) equation:

$$0 = \mu'_\alpha(\boldsymbol{r}^*) + \sum_\beta a'_{\alpha\beta}(\boldsymbol{r}^*)x_\beta, \tag{S.7}$$

In previous work [3], we derived two expressions which describe PCGRs in EO-type ecosystems under regimes in which the environment can change: the closed Environment-Organism (cEO) and fluctuating Environment-Organism (fEO) equations. While we previously derived these using a path integral approach, they can also be derived directly from the aEO equation. Integrating Eq. 5 yields the fEO equation

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = g_\alpha(\boldsymbol{r}_0) + \int_0^t \mu'_\alpha(\boldsymbol{r}) \cdot d\tau + \sum_\beta \int_0^t a'_{\alpha\beta}(\boldsymbol{r})x_\beta \cdot d\tau. \tag{S.8}$$

This describes the current growth rate of α based on the history of its interactions with other species and allogenic processes, along with an initial growth rate $g_\alpha(\boldsymbol{r}_0)$. In closed systems $\mu'_\alpha = 0$, yielding the cEO equation:

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = g_\alpha(\boldsymbol{r}_0) + \sum_\beta \int_0^t a'_{\alpha\beta}(\boldsymbol{r})x_\beta \cdot d\tau. \tag{S.9}$$

This family of equations represents different experimental scenarios. The cEO equation describes the dynamics of closed well-mixed systems (*e.g.* batch culture) and spatially structured systems under one dimensional flow (*e.g.* the lower gut) [3]. The fEO and aEO equations in turn describe open systems undergoing changes in environmental composition over time (*e.g.* seasonal variations in a lake), while the eEO equation captures open systems at equilibrium (*e.g.* a stable bioreactor). When a community with a given set of impacts and sensitivities is placed in these different contexts, we observe distinct but related dynamical outcomes. These equations allow us to formally relate these different outcomes to each other.

2.2 Supplementary Note 2: Comparison of separation of timescales and the accelerational mapping

Obtaining a general mapping between mechanistic theories and direct interaction models has been a long sought-after goal of theoretical ecology, as it would allow us to cast the interaction structure of ecosystems in terms of underlying biological mechanisms [4]. The standard way of approaching the problem is to assume a separation in the timescales of the environmental and population dynamics [5], as first introduced by MacArthur [6]. In brief, environmental variables are forced to equilibrate by setting Eq. 2 to zero, resulting in an algebraic constraint which can be substituted into Eq. 3. The construction of MacArthur's model ensures that the resulting composite coefficients can be mapped onto those of the gLV equation as constants.

While influential, this approach to the problem is brittle. Firstly, there is generally no natural separation between the two timescales, with environmental dynamics occurring at approximately the same rate as the population dynamics. Even more problematically for the study of systems fueled by abiotic resources, the mapping is not possible unless the environmental variables themselves have logistic dynamics (*i.e.* act as biotic resources with density-dependent reproduction rates) [4]. Separation of timescales is additionally used in other theoretical studies that do not assume logistic resource dynamics (*e.g.* [7]). However, the resulting models cannot be framed in terms of the gLV equation.

Our accelerational approach avoids these issues, providing a general and flexible mapping between the two frameworks. We can illustrate the differences between the two approaches by comparing the output of the EO mapping to MacArthur's original results. MacArthur's consumer-resource model can be written as:

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = \left(\sum_\rho w_\rho p_{\rho\alpha} r_\rho\right) - m_\alpha, \tag{S.10a}$$

$$\frac{dr_\rho}{dt} = \frac{r_\rho \nu_\rho}{k_\rho}(k_\rho - r_\rho) - r_\rho \sum_\beta p_{\rho\beta} x_\beta, \tag{S.10b}$$

where k_ρ is the carrying capacity for biotic resource ρ, $p_{\rho\alpha}$ represents the rate at which ρ is consumed by species α and converted to its own biomass, w_ρ is the nutritional value of ρ, ν_ρ is the maximal rate of increase of ρ and m_α is α's mortality rate (note that terms representing biomass conversion efficiency have been set to unity to simplify the expressions while retaining the structure of the model).

Separation of timescales yields the phenomenological gLV parameters [6]

$$a_{\alpha\beta} = -\sum_\rho p_{\rho\alpha} p_{\rho\beta} w_\rho \frac{k_\rho}{\nu_\rho}, \tag{S.11a}$$

$$\mu_\alpha = \sum_\rho k_\rho w_\rho p_{\rho\alpha} - m_\alpha. \tag{S.11b}$$

To apply our timescale separation-free approach, we begin by writing equation S.11b in terms of impact, sensitivity and allogenic functions (Table S1). Application of the definitions of μ'_α and $a'_{\alpha\beta}$ yields

$$a'_{\alpha\beta} = -\sum_{\rho} p_{\rho\alpha} p_{\rho\beta} w_\rho r^*_\rho, \tag{S.12a}$$

$$\mu'_\alpha = \sum_{\rho} \frac{r^*_\rho \nu_\rho}{k_\rho} (k_\rho - r^*_\rho) w_\rho p_{\rho_\alpha}. \tag{S.12b}$$

As the PCGA-based parameterisation depends on the equilibrium environment $\boldsymbol{r}^*$, we must also identify this composition. In this case, this is given by the intersection of Zero Net Growth Isoclines (ZNGIs) where $g_\alpha = 0$ for all species [8, 9]. Assuming the number of resources is equal to the number of species, this occurs at $\boldsymbol{r}^* = \boldsymbol{m}M^{-1}$, where the elements of the matrix M are $M_{\rho\alpha} = w_\rho p_{\rho\alpha}$ and $\boldsymbol{m}$ is the vector of mortalities m_α.

Comparing the two results, we note that the units of $a_{\alpha\beta}$ and μ_α are $[\text{Time}]^{-1}[\text{Biomass}]^{-1}$ and $[\text{Time}]^{-1}$ respectively, while those of $a'_{\alpha\beta}$ and μ'_α are $[\text{Time}]^{-2}[\text{Biomass}]^{-1}$ and $[\text{Time}]^{-2}$. The additional unit of inverse time in the accelerational quantities corresponds to the timescale of the resource dynamics, which is no longer eliminated through separation of timescales.

We also observe that the definitions of both $a_{\alpha\beta}$ and $a'_{\alpha\beta}$ consist of the sum of the products of the consumption rates $p_{\rho\alpha}$ $p_{\rho\beta}$ with a factor that only depends on the resource index ($w_\rho \frac{k_\rho}{\nu_\rho}$ for $a_{\alpha\beta}$ and $w_\rho r^*_\rho$ for $a'_{\alpha\beta}$). In Chesson's analysis of MacArthur's model [10], the $w_\rho \frac{k_\rho}{\nu_\rho}$s become the weights in a weighted least-squares regression between $p_{\rho\alpha}$ and $p_{\rho\beta}$, with the square root of the resulting coefficient of determination R^2 then being proportional to the niche overlap (see also Supplementary Note 2.3.3). The similarities in the structure of the expressions for $a_{\alpha\beta}$ and $a'_{\alpha\beta}$ means that this argument also applies to the acceleration-based analysis of MacArthur's model, allowing us to relate niche overlap to the linear independence of the utilisation vectors $\boldsymbol{p}_\alpha$ and $\boldsymbol{p}_\beta$.

2.3 Supplementary Note 3: Equivalencies between rate-based and accelerational ecological frameworks

In the main text, we note that the eEO equation (Eq. S.7) is structurally equivalent to the gLV equation at equilibrium. In this Note, we discuss several concepts that can be translated between the two frameworks on the basis of this equivalence: carrying capacity, linear stability and the stabilising/equalising mechanisms of Modern Coexistence Theory (MCT).

2.3.1 Carrying capacity

In Lotka-Volterra type systems, the carrying capacity of a species K_α is defined as its equilibrium abundance in isolation:

$$\frac{1}{x_\alpha}\frac{dx_\alpha}{dt} = \mu_\alpha + a_{\alpha\alpha}x_\alpha \tag{S.13a}$$

$$\implies 0 = \mu_\alpha + a_{\alpha\alpha}K_\alpha \tag{S.13b}$$

$$\implies K_\alpha = -\frac{\mu_\alpha}{a_{\alpha\alpha}}. \tag{S.13c}$$

This captures the concept of self-limitation in ecosystems – assuming $a_{\alpha\alpha}$ is negative (due to self-competition) and μ_α is positive (*i.e* α can establish in the current environment in isolation), K_α is a positive, finite value that indicates the density of α at which competition with itself exactly balances its underlying ability to grow.

Similar logic can be applied to obtain an equivalent expression for our accelerational framework:

$$\frac{d}{dt}\left(\frac{1}{x_\alpha}\frac{dx_\alpha}{dt}\right) = \mu'_\alpha + a'_{\alpha\alpha}x_\alpha \tag{S.14a}$$

$$\implies 0 = \mu'_\alpha + a'_{\alpha\alpha}K_\alpha \tag{S.14b}$$

$$\implies K_\alpha = -\frac{\mu'_\alpha}{a'_{\alpha\alpha}} = -\frac{\nabla g_\alpha \cdot \boldsymbol{\sigma}}{\nabla g_\alpha \cdot \boldsymbol{f}_\alpha}. \tag{S.14c}$$

Though equivalent in structure, this equation gives us a deeper mechanistic understanding of self-limitation. For example, we can see that increasing the elements of $\boldsymbol{\sigma}$ representing input of beneficial environmental factors (for which $\nabla g_{\alpha\rho}$ is positive) and decreasing the input of harmful environmental factors (for which $\nabla g_{\alpha\rho}$ is negative) will generally increase the value of the numerator, raising the overall value of K_α. The denominator is a more complex structure, particularly as trade-offs may enforce correlations between the elements of the vectors ∇g_α and $\boldsymbol{f}_\alpha$. However, we may broadly expect that an increase in the uptake of beneficial factors and release of harmful factors by α will make the denominator more negative and so reduce the carrying capacity overall.

An interesting case arises when $\mu'_\alpha = 0$, as in closed ecosystems. While this superficially suggests that the carrying capacity is $K_\alpha = 0$, in many systems the intra-specific biotic acceleration $a'_{\alpha\alpha}$ is also equal to 0 at equilibrium. This may arise when the release of growth-promoting substrates is exactly balanced by their uptake, as for example during nutrient cycling by algae [11], or when impacts fall to zero, as at the end of a batch culture experiment once all nutrients have been depleted. Under these conditions the carrying capacity is no longer defined by Eq. S.14,

and is instead determined by the initial conditions of the system (*e.g.* the initial amount of available nutrients). This implies that we can obtain very limited information about underlying ecological processes from these steady-state abundances in closed systems [12].

2.3.2 Linear stability analysis

Linear stability analysis of dynamical systems proceeds by identifying equilibrium states of the system and finding the eigenvalues of the system's Jacobian matrix evaluated at those points. Eigenvalues with positive real part indicate that arbitrarily small perturbations from the equilibrium state will be amplified by internal dynamics, pushing the system away from that point and destabilising the system. For the gLV model, the equilibrium composition $\boldsymbol{x}^*$ is given by [13]

$$\boldsymbol{x}^* = -A^{-1}\boldsymbol{\mu}, \tag{S.15}$$

and the Jacobian J^* at this stationary point is given by

$$J^* = \begin{pmatrix} a_{11}\,x_1^* & a_{12}\,x_1^* & \dots & a_{1N}\,x_1^* \\ a_{21}\,x_2^* & a_{22}\,x_2^* & \dots & a_{2N}\,x_2^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}\,x_N^* & a_{N2}\,x_N^* & \dots & a_{NN}\,x_N^* \end{pmatrix} \tag{S.16}$$

$$= \mathrm{diag}(\boldsymbol{x}^*)A. \tag{S.17}$$

Performing equivalent analysis in mechanistic models proceeds by evaluating the Jacobian of the entire dynamical system, *i.e.* incorporating both population and environmental dynamics (*e.g.* [14], Appendix A of [15]). By concatenating our expressions for the population and environmental dynamics and assuming that all populations are at non-zero densities, we obtain

$$J^* = \left(\begin{array}{c|c} \frac{\partial}{\partial x_\beta}\left(\frac{dx_\alpha}{dt}\right) & \frac{\partial}{\partial r_\gamma}\left(\frac{dx_\alpha}{dt}\right) \\ \hline \frac{\partial}{\partial x_\beta}\left(\frac{dr_\rho}{dt}\right) & \frac{\partial}{\partial r_\gamma}\left(\frac{dr_\rho}{dt}\right) \end{array}\right) \text{ at } \boldsymbol{r}^*, \boldsymbol{x}^* \tag{S.18}$$

$$= \left(\begin{array}{c|c} 0 & \mathrm{diag}(\boldsymbol{x}^*)\nabla G \\ \hline F & \frac{\partial}{\partial r_\gamma}(\sigma_\rho + \sum_\beta \boldsymbol{f}_\beta\, x_\beta^*) \end{array}\right), \tag{S.19}$$

where F is a matrix consisting of all of the impact functions evaluated at $\boldsymbol{r}^*$, ∇G is the matrix of all sensitivity gradients evaluated at $\boldsymbol{r}^*$ and ρ and γ are indices for the environmental factors. We show the real part of the eigenvalues of this matrix for the two species and five species auxotrophy models in Figs. S4b and S6d, respectively.

To understand how this result relates to the gLV-based analysis, let us begin by assuming that the environmental dynamics do not depend on the values of the environmental factors within the neighbourhood of $\boldsymbol{r}^*$ (*i.e.* that $\frac{\partial}{\partial r_\gamma}(\sigma_\rho + \sum_\beta \boldsymbol{f}_\beta\, x_\beta^*) = 0$). This results in a 2×2 block matrix with blocks of zeros on the diagonal, for which the eigenvalues are given by $\pm\sqrt{\mathrm{eig}(\mathrm{diag}(\boldsymbol{x}^*)\nabla G F)} = \pm\sqrt{\mathrm{eig}(\mathrm{diag}(\boldsymbol{x}^*)A')}$. In other words, this simplifying assumption yields eigenvalues that are the square root of the eigenvalues s_k that we would obtain from a direct translation of Eq. S.17 from rate-based to accelerational quantities.

This result implies that combinations of A and $\boldsymbol{x}^*$ that produce at least one positive real eigenvalue in the gLV framework have corresponding A's and $\boldsymbol{x}^*$s in the simplified accelerational framework that have at least one positive real eigenvalue. This implies that linear instability in the gLV framework corresponds to linear instability in the accelerational framework. Perhaps more surprisingly, combinations of A and $\boldsymbol{x}^*$ that produce only negative *real* eigenvalues (*i.e.* are stable gLV systems) have corresponding A's and $\boldsymbol{x}^*$s which produce paired *imaginary* eigenvalues. These correspond to neutrally stable systems, in which perturbations from the equilibrium state result in indefinite oscillations around that equilibrium.

Clearly the environmental self-regulation represented by $\frac{\partial}{\partial r_\gamma}(\sigma_\rho + \sum_\beta \boldsymbol{f}_\beta\, x^*_\beta)$ is playing an important role in ensuring convergence on the system's fixed point. Let us incorporate these terms into our analysis more carefully. Following [16], we can use Schur's formula and note that the eigenvalues of a matrix and its transpose are the same to obtain the following defining equation for the full system's eigenvalues λ:

$$0 = \mathrm{det}(-\lambda I_N)\mathrm{det}\left(-\lambda I_N - \mathrm{diag}(\boldsymbol{x}^*)\nabla G\left[\frac{\partial}{\partial r_\rho}(\sigma_\gamma + \sum_\beta \boldsymbol{f}_\beta\, x^*_\beta) - \lambda I_M\right]^{-1} F\right), \tag{S.20}$$

where we denote the identity matrix of size N as I_N, N is the total number of populations and M is the total number of environmental factors. Assuming that the system is not neutrally stable ($\lambda \neq 0$), this becomes

$$0 = \mathrm{det}\left(-\lambda I_N - \mathrm{diag}(\boldsymbol{x}^*)\nabla G\left[\frac{\partial}{\partial r_\rho}(\sigma_\gamma + \sum_\beta \boldsymbol{f}_\beta\, x^*_\beta) - \lambda I_M\right]^{-1} F\right). \tag{S.21}$$

It is difficult to make general statements about this equation. However, it is informative to consider the case of a chemostat in which only dilution at rate D contributes to the environmental self-regulation (*i.e.* $\nabla \boldsymbol{f}_\beta = 0$). As $\frac{\partial \sigma_\gamma}{\partial r_\rho} = -D I_M$ in this case, we obtain

$$0 = \mathrm{det}\left(-\lambda I_N - \mathrm{diag}(\boldsymbol{x}^*)\nabla G\left[(-D-\lambda)I_M\right]^{-1} F\right) \tag{S.22a}$$

$$= \mathrm{det}\left(\lambda(\lambda + D)I_N - \mathrm{diag}(\boldsymbol{x}^*)A'\right). \tag{S.22b}$$

Referring to these eigenvalues of $\mathrm{diag}(\boldsymbol{x}^*)A'$ as s_k, we can write the eigenvalues λ_k of the dilution-only EO system as

$$\lambda_k(\lambda_k + D) = s_k, \tag{S.23}$$

with solution

$$\lambda_k = -\frac{D}{2} \pm \sqrt{\frac{D^2}{4} + s_k}. \tag{S.24}$$

The upshot of this analysis is that chemostat-type dilution of the environmental factors tends to stabilise the ecosystem. The term outside the square root ($-\frac{D}{2}$) is always real and negative,

thus damps down the neutral oscillations that would result if the expression under the square root remained negative.

From these considerations, we can propose conditions for stability of Eq. S.19: firstly, the accelerational version of Eq. S.17 should have negative real eigenvalues to ensure neutral stability in the absence of environmental self-regulation. Secondly, environmental self-regulation should be dissipative (*e.g.* through environmental dilution) to dampen the resulting oscillations. Unfortunately, comparison of the eigenvalue spectra of the accelerational version of Eq. S.17 (Fig. S5a) and Eq. S.19 (Fig. S5b) for the two species amino acid production sweep shows this picture is overly simplistic. While the eigenvalues of the former are predominantly real and negative, we observe complex eigenvalues with negative real part and even positive real eigenvalues at the lowest production rates. Given that we observe stability in the full system, the environmental self-regulation must be doing more than merely damping neutral oscillations. Nonetheless, the similarity in the structure of the two sets of eigenvalues is striking and suggests that the eigenvalues of $\mathrm{diag}(\boldsymbol{x}^*)A'$ may be a good proxy for evaluating system stability under a broad range of contexts. Exploring the deeper connections between these two perspectives under a wider range of scenarios should be a focus of future work.

2.3.3 Modern Coexistence Theory

Modern Coexistence Theory (MCT) arose as an effort to consolidate coexistence mechanisms varying across scales and systems into a limited set of processes operating on similar mathematical principles [17, 18]. One of its most important conclusions is that influences on species coexistence can be mathematically partitioned into equalising mechanisms (processes that reduce the difference in average fitness between species) and stabilising mechanisms (processes that reduce the overlap of niches between species) [17]. We will focus on the two-species case here, noting in passing that the multi-species case can be investigated through other frameworks that may in principle be translated into our accelerational formulation through Eq. 6 [19].

Much of our understanding of MCT comes from two-species gLV systems. For competing species labelled 1 and 2, feasibility is determined by the solutions of the equations $x_1^* = \frac{a_{22}\mu_1 - a_{12}\mu_2}{a_{11}a_{22}-a_{21}a_{12}}$ and $x_2^* = \frac{a_{11}\mu_2 - a_{21}\mu_1}{a_{11}a_{22}-a_{21}a_{12}}$. Combining the feasibility requirements of both species implies that

$$\frac{a_{12}}{a_{22}} < \frac{\mu_2}{\mu_1} < \frac{a_{11}}{a_{21}} \tag{S.25}$$

must hold for mutual coexistence.

One of Chesson's insights was to show that the expression $\frac{a_{\alpha\beta}}{a_{\alpha\alpha}}$ also appears as the regression coefficient between the resource preference vectors in MacArthur's consumer-resource model (Supplementary Note 2.2, [10]), allowing this expression describing feasible coexistence to be formally related to the degree of similarity between the utilisation of resources by the two populations. This led to a form of the feasibility criterion in which each component has been multiplied by $\sqrt{\frac{a_{22}a_{21}}{a_{11}a_{12}}}$:

$$\sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}} < \frac{\mu_1}{\mu_2}\sqrt{\frac{a_{22}a_{21}}{a_{11}a_{12}}} < \sqrt{\frac{a_{11}a_{22}}{a_{12}a_{21}}}. \tag{S.26}$$

This expression states that the 'fitnesses ratio' $\frac{\mu_2}{\mu_1}\sqrt{\frac{a_{22}a_{21}}{a_{11}a_{12}}}$ must be between the 'niche overlap' $\sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}}$ and its reciprocal for coexistence to be feasible. We must also account for stability,

which is guaranteed if $a_{22}a_{11} > a_{21}a_{12}$. Thus, stable coexistence results provided both that Eq. S.26 holds and the niche overlap term $\sqrt{\frac{a_{12}a_{21}}{a_{11}a_{22}}}$ is less than 1. This latter condition is essentially always met as self-limitation is stronger than partner suppression (*i.e.* a_{11} is more negative than a_{21} and a_{22} is more negative than a_{12}) under competition scenarios.

It should be emphasised that Eq. S.26 is not MCT in and of itself, but rather the result of applying MCT to the two-species gLV equation as a specific ecosystem model [18]. Alternative definitions of niche overlap and fitness differences have been proposed for more general models [19–21], although the compatibility of some of these alternative definitions with those of Eq. S.26 has been called into question [22].

The ability to write MacArthur's consumer-resource model in gLV form using separation of timescales (Section 2.2) has led to its widespread adoption as a basis for interpreting niche overlaps and fitness differences in mechanistic terms [9, 10, 22]. Our mapping between mechanistic models and the gLV equation (Eq. 6) allows us to generalise this approach for any mechanistic model by substituting in our effective gLV parameters evaluated at equilibrium:

$$\sqrt{\frac{a'_{12}a'_{21}}{a'_{11}a'_{22}}} < \frac{\mu'_2}{\mu'_1}\sqrt{\frac{a'_{22}a'_{21}}{a'_{11}a'_{12}}} < \sqrt{\frac{a'_{11}a'_{22}}{a'_{12}a'_{21}}}. \tag{S.27}$$

We discuss the application of this result to the two species auxotrophy model in the main text.

One downside of this formulation is that it loses its connection to the mutual invasibility criterion, which states that both species must be able to invade the other when inoculated at low density for them to stably coexist. This is because differing equilibrium environmental states may mean that the effective gLV parameters for a resident community with $N-1$ species are very different to those of the full N species community. However, the mutual invasibility criterion is known to have a number of deficiencies as a predictor of coexistence in mechanistic models [23]. In the case of our own model, it is impossible for either of the obligate mutualists to reach an equilibrium in the absence its partner (Fig. S1), meaning there is no such thing as a baseline single-species community which can be subjected to invasion. Our approach may provide further insights into the conditions under which the mutual invasibility criterion breaks down as a predictor of coexistence.

3 Supplementary Tables

Simulated system	Sensitivity functions	Impact functions	Allogenic function
Abiotic nutrient competition [7]	$g_\alpha = \sum_\rho b_\rho c_{\alpha\rho} m_\alpha(r_\rho) - \theta$	$f_{\alpha\rho} = -c_{\alpha\rho} m_\alpha(r_\rho)$	$\sigma_\rho = r_{\rho\ \mathrm{IN}} - d_\rho r_\rho$
	b_ρ = nutritional value of ρ, $c_{\alpha\rho}$ = consumption rate of ρ by α, d_ρ = degradation rate.		
Antibiotic-mediated coexistence [24]	$g_\alpha = m(r_4) e^{-b_\alpha r_\alpha} - \theta_\alpha$	$f_{\alpha\rho} = \begin{cases} c m_\alpha(r_4) & \text{if } \rho = \alpha+1 \text{ (ρ is secreted antibiotic)} \\ -d r_\rho & \text{if } \rho = \alpha+2 \text{ (ρ is degraded antibiotic)} \\ -m_\alpha(r_\rho) & \text{if } \rho = 4 \text{ (ρ is nutrient)} \\ 0 & \text{otherwise} \end{cases}$	$\boldsymbol{\sigma} = D(\boldsymbol{r}_{\mathrm{IN}} - \boldsymbol{r})$
	b_α = antibiotic toxicity, c = per-capita antibiotic production rate, d = antibiotic degradation rate. $\boldsymbol{r}$: position α = α-targeting antibiotic, $\alpha+1$ = antibiotic secreted by α, $\alpha+2$ = antibiotic degraded by α, 4 = in-common nutrient.		
Nutrient cross-feeding [25]	$g_\alpha = \sum_\rho b_{\alpha\rho} c_{\alpha\rho} r_\rho - \theta_\alpha$	$f_{\alpha\rho} = -c_{\alpha\rho} r_\rho + \sum_\gamma d_{\alpha\rho\gamma} c_{\alpha\gamma} r_\gamma$	$\boldsymbol{\sigma} = D(\boldsymbol{r}_{\mathrm{IN}} - \boldsymbol{r})$
	$b_{\alpha\rho}$ = biomass conversion efficiency, $c_{\alpha\rho}$ = uptake rate of ρ by α, $d_{\alpha\rho\gamma}$ = stoichiometry matrix.		
Chemical-mediated interactions [26]	$g_\alpha = b_\alpha + \sum_\rho (m_\alpha^+(r_\rho) + c_{\alpha\rho}^- r_\rho)$	$f_{\alpha\rho} = d_{\alpha\rho} - o_{\alpha\rho} m_\alpha(r_\rho)$	$\boldsymbol{\sigma} = \mathbf{0}$
	$b_{\alpha\rho}$ = basal growth rate, $c_{\alpha\rho}$ = growth impact of ρ on α, $d_{\alpha\rho}$ = production rate of ρ by α, $o_{\alpha\rho}$ = yield constant.		
Tilman's consumer-resource model [8]	$g_\alpha = m_\alpha(\sum_\rho b_{\alpha\rho} - c_\alpha) - D$	$f_{\alpha\rho} = -d_{\alpha\rho}$	$\boldsymbol{\sigma} = D(\boldsymbol{r}_{\mathrm{IN}} - \boldsymbol{r})$
	$b_{\alpha\rho}$ = value of ρ for α, c_α = energy allocated to α's maintenance, $d_{\alpha\rho}$ = consumption rate of ρ by α.		
MacArthur's consumer-resource model [6]	$g_\alpha = \sum_\rho b_\rho c_{\alpha\rho} r_\rho - \theta_\alpha$	$f_{\alpha\rho} = -c_{\alpha\rho} r_\rho$	$\sigma_\rho = \frac{r_\rho d_\rho}{o_\rho}(o_\rho - r_\rho)$
	b_ρ = nutritional value of ρ, $c_{\alpha\rho}$ = consumption rate of ρ by α, d_ρ = maximal reproduction rate of ρ, o_ρ = carrying capacity of ρ.		

Table S1: **Non-exhaustive list of established EO models.** b, c, d and o are model-specific parameters whose meanings are given below the model specification. $m_\alpha(r_\rho)$ denotes a Monod function acting on resource r_ρ with embedded half-velocity and maximal utilization rate constants. θ denotes a mortality rate, while D denotes a dilution rate. Superscripts $^+$ and $^-$ indicate that the associated terms only apply if they are positive or negative, respectively. ρ and γ are indices of the environment vector $\boldsymbol{r}$. Where necessary, the structure of the environment vector with respect to the species index α is also specified. Note that in some cases, indices in $\boldsymbol{r}$ are modulo the length of the environment vector (for example, index $\alpha+1$ in the antibiotic-mediated coexistence model is written fully as $(\alpha+1) \mod 3$). We suppress the modulo operation in these cases for notational compactness. Some additional constraints on parameters present in the original models (*e.g.* enforcing conservation of mass) are not specified. Please see cited publications for complete model specifications including parameter choices.

Parameter symbol	Parameter meaning	Value
$\nu_{\mathring{v}}$	Maximum growth rate of $\mathring{v}$	0.4
$\nu_{\mathring{w}}$	Maximum growth rate of $\mathring{w}$	0.4
$K_{\mathring{v}\ c}$	Half velocity constant for use of c by $\mathring{v}$	1.8
$K_{\mathring{v}\ v}$	Half velocity constant for use of v by $\mathring{v}$	0.8
$K_{\mathring{w}\ c}$	Half velocity constant for use of c by $\mathring{w}$	0.7
$K_{\mathring{w}\ w}$	Half velocity constant for use of w by $\mathring{w}$	1.4
$Y_{\mathring{v}\ c}$	Yield of $\mathring{v}$ from one unit of c	1
$Y_{\mathring{v}\ v}$	Yield of $\mathring{v}$ from one unit of v	5
$Y_{\mathring{w}\ c}$	Yield of $\mathring{w}$ from one unit of c	1
$Y_{\mathring{w}\ w}$	Yield of $\mathring{w}$ from one unit of w	10
ℓ_v	Production rate of v by $\mathring{w}$	0.025
ℓ_w	Production rate of w by $\mathring{v}$	0.05
$[c]_{\mathsf{IN}}$	Input concentration of carbon source	2
$[v]_{\mathsf{IN}}$	Input concentration of amino acid v	0
$[w]_{\mathsf{IN}}$	Input concentration of amino acid w	0

Table S2: **Parameters for the two-species auxotrophy model.** Note that values of ℓ_v, ℓ_w, $[v]_{\mathsf{IN}}$ and $[w]_{\mathsf{IN}}$ are modified in some figures. These modifications are specified in the figure caption or axes labels.

4 Supplementary Figures

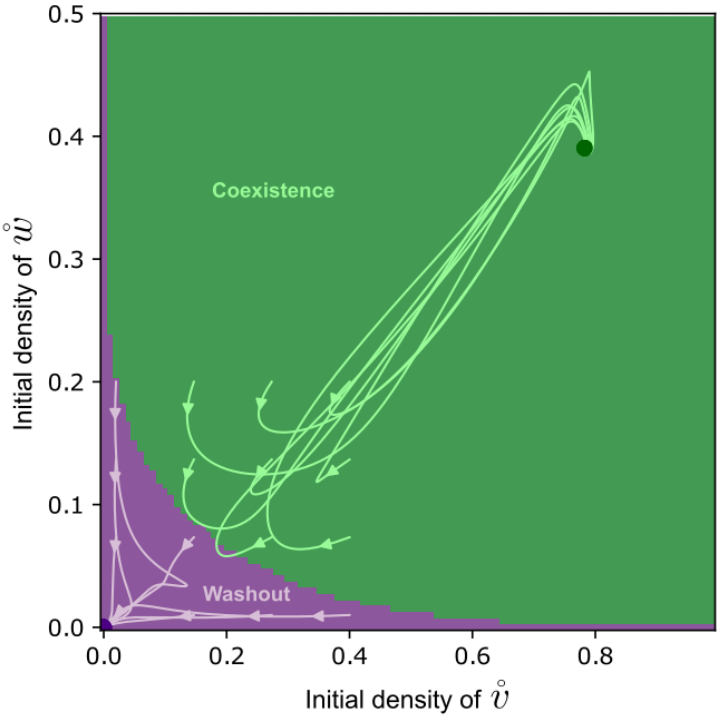

Figure S1: **Coexistence and Allee effects in the cross-feeding model.** Coloured sectors indicate basins of attraction for different starting densities for the two auxotrophs (green, mutual coexistence, purple, mutual extinction). Example trajectories converging on the two attractors (purple point and green point) are also shown. For a given starting density of one of the auxotrophs, pushing the starting density of the partner down to sufficiently low levels results in mutual extinction, an example of an Allee effect. This is a characteristic of obligate mutualisms [27]. Input media contains only the carbon source (no amino acids). $D = 0.01$.

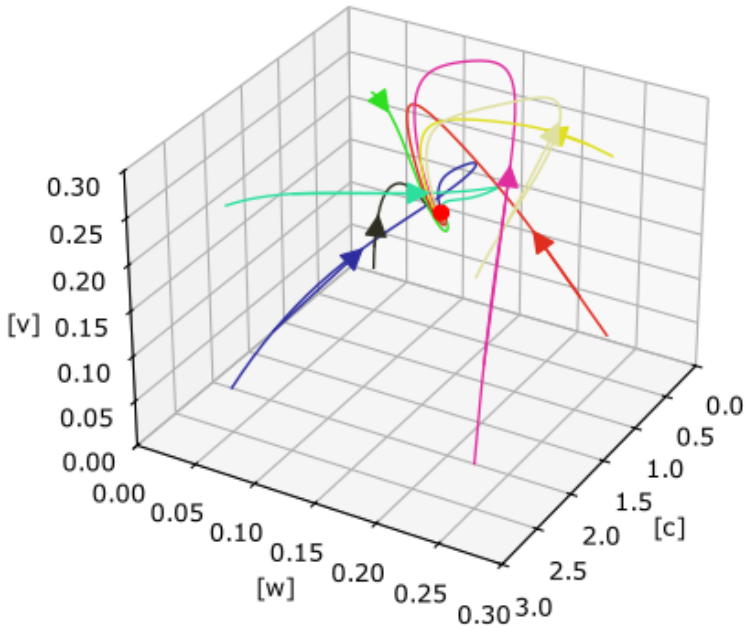

Figure S2: **The three-dimensional environment space.** Environmental trajectories for two-species chemostats initiated with varying initial compositions are plotted in different colours and converge on the equilibrium state $\boldsymbol{r}^*$ (red dot). Initial concentrations of the amino acids v and w are 0.05 or 0.25, while those of the carbon source c are 0.5 or 2.5. Input media contains only carbon at $[c]_{\text{IN}} = 3$. $D = 0.01$.

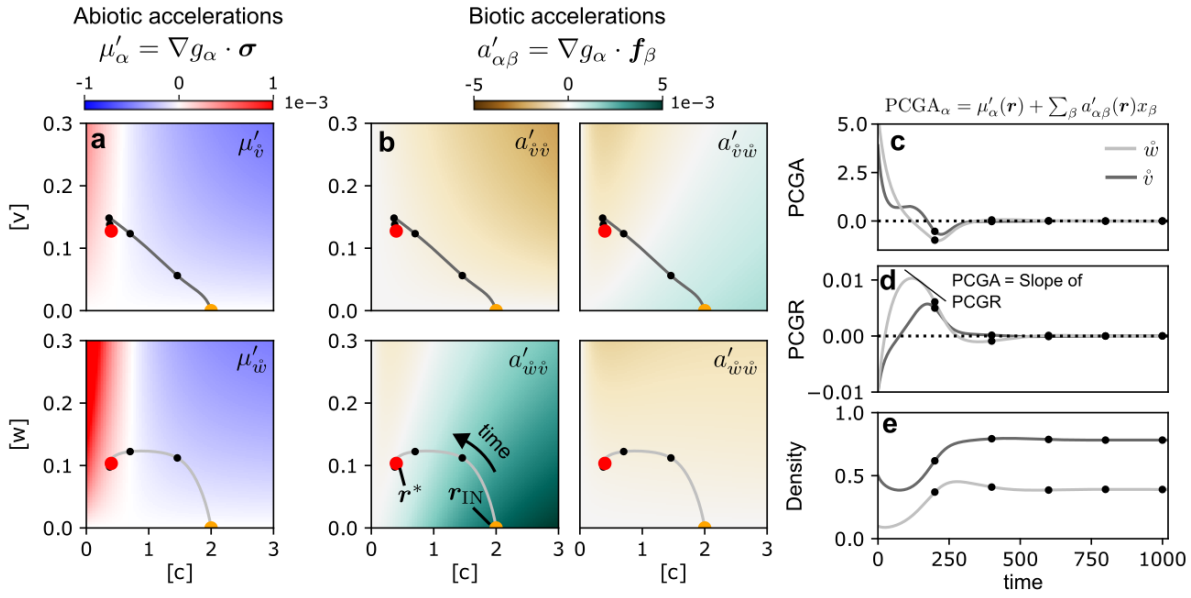

Figure S3: **The aEO equation describes ecosystem transients.** a,b) Scalar fields of the abiotic accelerations (a) and biotic accelerations (b), replicated from Fig. 2. We plot on top of these an environmental trajectory for a system initiated with the same environment as in the input media, which contains only carbon. The per-capita growth acceleration (PCGA) is determined by the aEO equation (Eq. 5), which is parameterised using the value of these scalar fields at each timepoint (c). Integration of the PCGA gives the time-varying per-capita growth rate (PCGR), which can itself be integrated to give the dynamics of the population densities (e). Black points along environmental trajectories (a,b) correspond to equally-spaced timepoints (c-e). $D = 0.01$.

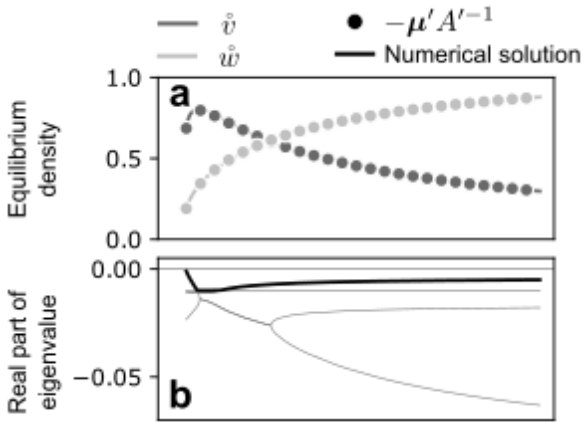

Figure S4: **Equilibrium abundances and stability analysis for amino acid production sweep.** Shown are equilibrium abundances (a) and the real part of the eigenvalue spectrum (b) (Supplementary Note 2.3.2, Fig. S5) for the set of simulations presented in Fig. 3. We also confirmed the equivalency between Eqs. 6 and 1 by using the values of the effective gLV parameters to estimate the equilibrium densities through the expression $\boldsymbol{\mu}' A'^{-1}$, where $\boldsymbol{\mu}'$ and A' are built from the equilibrium values shown in panels a and d of Fig. 3 (a, circles).

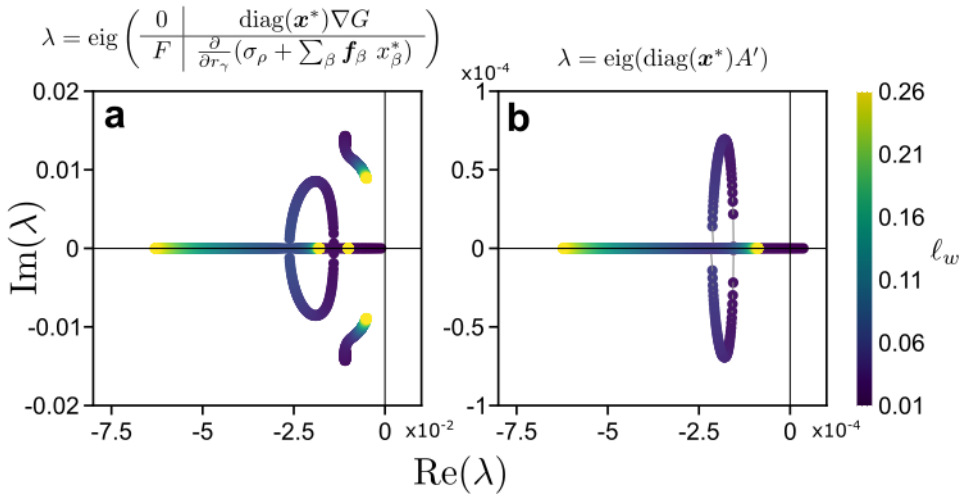

Figure S5: **Eigenvalue spectra for mechanistic models.** We show variations in the eigenvalue spectrum of the two-species auxotrophy model as the production rate of w by $\mathring{v}$ (ℓ_w) is varied. In a), we show the eigenvalues of the true Jacobian matrix which includes resource dynamics (Eq. S.19). In b), we show the eigenvalues of the matrix which is the accelerational analogue of the gLV Jacobian (Eq. S.17). The simulations used to construct this figure are the same as those used in Fig. 3. Note that we show the real part of panel a in Fig. S4b.

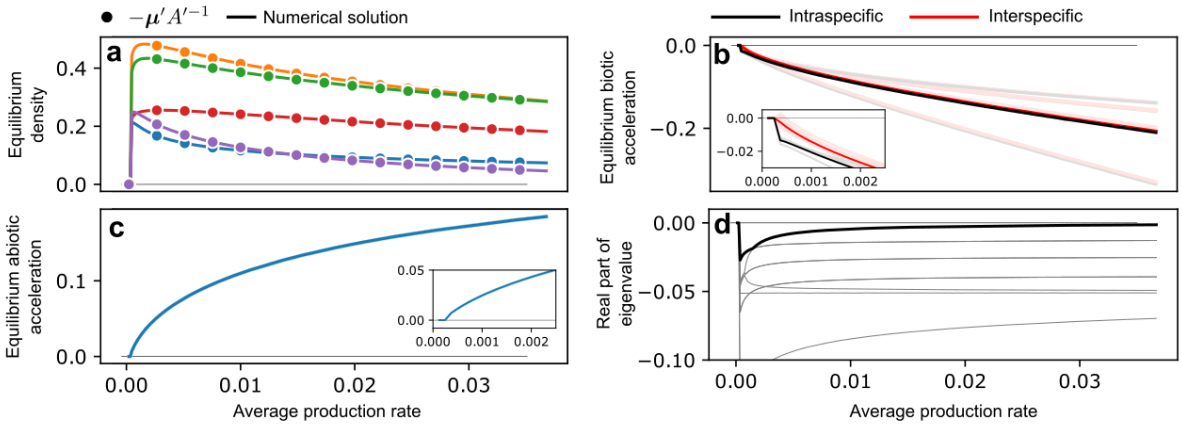

Figure S6: **5 species cross-feeding communities can be stable.** We constructed simulated communities of five auxotrophs with randomised mechanistic parameters and grew them in a chemostat on amino acid-free media (Methods). As in Fig. 3, we show the equilibrium abundances (a), biotic accelerations (b, red lines indicate interspecific values and black lines intraspecific values), abiotic accelerations (c) and real part of eigenvalue spectrum (d) for different average amino acid production rates. Insets in panels b and c show zoomed-in regions of the main plot. Black line in d indicates value of the maximal eigenvalue.

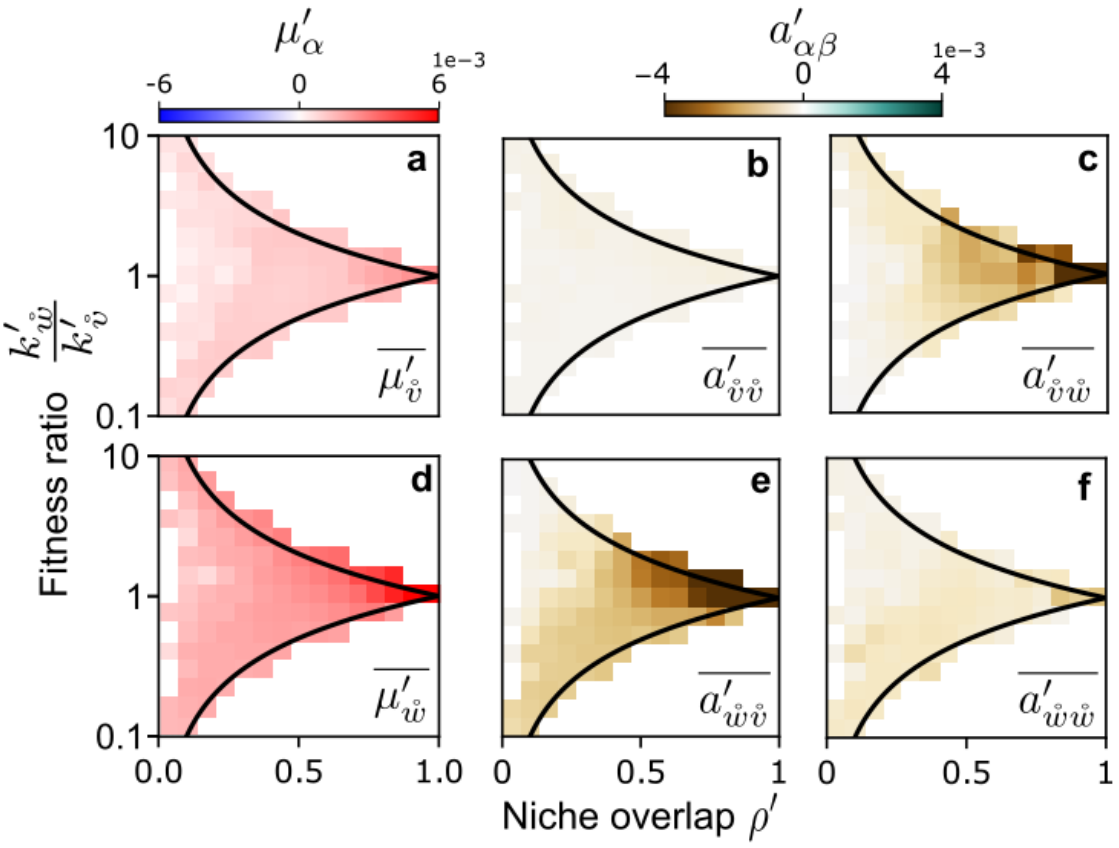

Figure S7: **Variation of effective gLV parameters on the niche overlap and relative fitness difference axes.** We display equivalent analysis as in Fig. 5a-g for the effective gLV parameters of the two species auxotrophy model. Shown are averages of abiotic accelerations (a,d) and biotic accelerations (b,c,e,f).

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